

The Distribution of the Sample Minimum-Variance Frontier

Raymond Kan

Joseph L. Rotman School of Management, University of Toronto, Toronto,
Ontario M5S 3E6, Canada, kan@chass.utoronto.ca

Daniel R. Smith

Faculty of Business Administration, Simon Fraser University, Burnaby,
British Columbia V5A 1S6, Canada, drsmith@sfu.ca

In this paper, we present a finite sample analysis of the sample minimum-variance frontier under the assumption that the returns are independent and multivariate normally distributed. We show that the sample minimum-variance frontier is a highly biased estimator of the population frontier, and we propose an improved estimator of the population frontier. In addition, we provide the exact distribution of the out-of-sample mean and variance of sample minimum-variance portfolios. This allows us to understand the impact of estimation error on the performance of in-sample optimal portfolios.

Key words: minimum-variance frontier; efficiency set constants; finite sample distribution

History: Accepted by David A. Hsieh, finance; received April 4, 2007. This paper was with the authors 2 weeks for 1 revision. Published online in *Articles in Advance* May 22, 2008.

1. Introduction

The minimum-variance frontier plays a central role in much of finance. For example, it is the cornerstone of modern portfolio theory and many asset pricing models. In addition, it is closely linked to the Hansen-Jagannathan bound on the variance of admissible stochastic discount factors (Hansen and Jagannathan 1997). It is a common practice in the finance literature to trace out the minimum-variance frontier using sample estimates of the mean and covariance matrix of returns. However, a sample frontier is subject to estimation error, so it is important to understand its finite sample distribution.

While the sample frontiers are widely calculated and interpreted, their finite sample properties are virtually unknown. Dickinson (1974) shows that the sample variance of the sample global minimum-variance portfolio has a chi-squared distribution.¹ For the two assets case, Dickinson (1974) also provides the exact distribution of the weights of the sample global minimum-variance portfolio. Jobson and Korkie (1980) provide the exact mean and variance of the three sample efficiency set constants that determine the sample minimum-variance frontier. In addition, they provide approximation formulas for the mean and variance of the weights of the sample tangency portfolio. Jobson (1991) provides the exact

distribution for two out of the three random variables that are crucial in determining the location of the sample minimum-variance frontier. Okhrin and Schmid (2006) derive the exact mean and variance of the weights of the sample optimal portfolio that maximizes a given expected quadratic utility function. For the special case of sample global minimum-variance portfolio, they also provide the exact distribution of its weights.

We contribute to this literature by deriving the finite sample distribution and moments of the sample minimum-variance frontier when returns are independent multivariate normal random variables. We find that the sample minimum-variance frontier is a heavily biased estimator of the population frontier, even when the length of the estimation window is very long. To correct for this bias, we propose a new adjusted estimator of the population frontier that has a significantly smaller bias than the traditional sample estimator.

For many investors, inference on the population minimum-variance frontier is of little interest when they cannot hold the portfolios on the population frontier because the mean and covariance matrix of the returns are unknown. The classical approach to mean-variance portfolio selection (Markowitz 1952) involves estimating the sample mean and covariance matrix of returns, and then treating these as their population values when selecting the optimal portfolio. However, using sample estimates of the

¹Dickinson's (1974) result contains an error in the degrees of freedom.