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PREPAYMENT RISK AND THE DURATION OF DEFAULT-FREE MORTGAGE-BACKED SECURITIES

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Abstract

The conventional duration measure for mortgage-backed pass-through securities assumes that the prepayment rate is invariant to changes in market interest rates. In this paper, the conventional duration is modified to take into account the interest-rate sensitivity of mortgage prepayments. Including interest rate sensitivity is shown to reduce substantially the duration of a mortgage-backed pass-through security when the current mortgage rate is less than the contract rate.

I. Introduction

The conventional duration measure for a mortgage or mortgage-backed pass-through security (MBS) is based on the assumption that the probability of prepayment is independent of changes in the market interest rate (Kolb, Corgel, and Chiang (1982), Anderson and Chiang (1987), Financial GNMA Pass-Through Yield and Value Tables (1988)). Navratil (1985), Green and Shoven (1986), Heuson (1988), and others show that the prepayment experience of mortgage pools with similar intrinsic characteristics is related inversely to the difference between the current rate and the contract rate of the pool.[1] As the current rate declines, prepayment rates increase and the expected life of the loan decreases. Excluding this inverse relationship between changes in the current rate and the prepayment rates biases the conventional duration measure.

The conventional duration measure is further limited because it is formulated in terms of changes in the cash-flow yield of the MBS.[2] At any level of current mortgage rate, the cash-flow yield on seasoned issues with different contract rates will be different, because prepayment expectations and prepayment risk depend upon the relation between the current rate and the contract rate. Therefore, conventional duration measures formulated in terms of cash-flow yields are not comparable across seasoned issues with different contract rates. This limitation can be overcome by defining the duration in terms of an interest rate that is not specific to any given MBS.

In this paper the duration for an MBS is specified in terms of the current rate rather than the cash-flow yield. The advantages of this formulation are twofold: (a) the duration measure for an MBS is defined with respect to a common numeraire, the current rate; and (b) the numeraire, unlike the cash-flow

[1]In this paper the current rate denotes the market rate for newly issued mortgages selling at par, and the contract rate denotes the rate at which the mortgage was amortized at the time of inception.

[2]The cash-flow yield is the internal rate of return of the MBS based on the expected cash flows.

yield, is directly observable and invariant to the specification of the prepayment rate function.

II. Modified Duration for Mortgage-Backed Securities

The market value of an MBS may be defined in terms of the present value of expected future cash flows:

$$M_0 = \sum_{t=1}^{T} \frac{E(C_t|i)}{(1 + y)^t} \tag{1}$$

where

$E(C_t|i)$ = the expected value of cash-flow C in period t conditional on the current rate i;
y = the cash-flow yield; and
T = the number of contracted payments in the mortgage.

Most of the analysis of the prepayment history of MBS is conducted on pools of mortgages. A convention often used by practitioners in evaluating prepayment activity is to assume a constant prepayment rate (CPR) for a pool of mortgages.[3] This method assumes that the prepayment rate remains the same for each period over the life of a mortgage. The CPR is defined as the percentage of the remaining balance of the pool, adjusted for past prepayments[4] and normal amortization, which is paid off in a year.

From knowledge of the CPR, the original value of the pool, and the contract rate, the projected cash flows for a mortgage pool may be determined. Define ϕ as the constant monthly prepayment rate derived from the CPR,[5] A as the scheduled interest and principal payment based on the value of the pool at inception, B_t^* as the amortized value (assuming zero prepayment) of the original value of the pool at the contract rate i_0 at month t, and S_t as the remaining (or surviving) balance of the pool at month t.

The constant monthly prepayment rate is defined (see Bartlett (1989, pp. 167–69)) as follows:

[3]Another widely adopted convention is the Public Security Association (PSA) model. It assumes that an MBS prepays at a successively increasing rate for the first thirty months and then at a constant rate thereafter (see Bartlett (1989, 167–72)) . Thus, for mortgages seasoned for more than thirty months the two methods are equivalent. The CPR for a seasoned mortgage equals the PSA multiplied by .06. For example, 100 percent PSA corresponds to 6 percent CPR.

[4]Past prepayments consist of full and partial prepayments made on mortgages contained within the pool.

[5]The CPR is defined on an annual basis. The monthly constant prepayment rate ϕ equals $1 - (1-\text{CPR})^{1/12}$.

$$\phi = 1 - \frac{S_t/B_t^*}{S_{t-1}/B_{t-1}^*} \tag{2}$$

Solving equation (2) for the remaining balance at month t gives,

$$S_t = (1 - \phi)\left(\frac{B_t^*}{B_{t-1}^*}\right)S_{t-1}, \text{ for } t = 1, 2, \ldots, T \tag{3}$$

From the initial condition $S_0 = B_0^*$, equation (3) may be solved recursively, yielding

$$S_t = (1 - \phi)^t B_t^* \tag{4}$$

The balance of the pool at month t after accounting for amortization and prepayment is S_t. The balance at month t before accounting for prepayment is $S_t = (1 - \phi)^{t-1}B_t^*$. This latter formula is important in determining the projected prepayment at month t.

In any given month the projected cash flow can be broken down into two components: (a) the scheduled payment of interest and principal on the remaining balance of the loan and (b) the partial or full prepayment of mortgages contained in the pool. The projected scheduled prepayment in month t is based on the remaining balance at month $t - 1$. It equals the original monthly payment A scaled down by the ratio of the remaining balance to the amortized value of the original pool at month $t - 1$: S_{t-1}/B_{t-1}^*. Thus, the projected scheduled payment equals $(1 - \phi)^{t-1}A$. The projected prepayment at month t equals the prepayment rate multiplied by the balance of the pool before prepayment: $\phi(1 - \phi)^{t-1}B_t^*$.

The projected cash flow at month t equals the sum of the projected scheduled payments and prepayments:[6]

$$E(C_t) = A(1 - \phi)^{t-1} + B_t^*\phi(1 - \phi)^{t-1} \tag{5}$$

Defining $B_t = B_t^* + A$, equation (5) may be written in a more convenient form:

$$E(C_t) = A(1 - \phi)^t + B_t\phi(1 - \phi)^{t-1} \tag{6}$$

[6]One can verify the validity of this formulation of the cash flows for a pool with a constant prepayment rate by observing that the projected balance in month t equals the preceding balance plus accrued interest minus the projected cash flow, as expressed in equation (5), at month t:

$$S_t = B_{t-1}^*(1 - \phi)^{t-1}(1 + i_0) - A(1 - \phi)^{t-1} - B_t^*\phi(1 - \phi)^{t-1}$$

Since

$$B_t^* = B_{t-1}^*(1 + i_0) - A,\ S_t = (1 - \phi)^t B_t^*$$

which is given by equation (4).

Note that equation (6) has the same form as the expression for the expected cash flow from a single mortgage if prepayment in one month is independent of the same event in another month, and if partial prepayments are disallowed. Define $\hat{\phi}$ as the monthly probability of prepayment, $\hat{A}$ as the scheduled payment of interest and principal, and $\hat{B}_t$ as the balance of the mortgage at period t, which equals the present value at the contract rate of the remaining scheduled payments plus scheduled payment $\hat{A}$. At any given month t, there are three possible outcomes: the mortgage survives to month t, the mortgage is prepaid in month t, or the mortgage was prepaid in a previous month. The cash flow associated with the mortgage surviving to month t is $\hat{A}$ and the probability of this outcome is $(1 - \hat{\phi})^t$. The cash flow associated with prepayment in month t is the balance of the loan at month t, $\hat{B}_t$. The probability of this outcome equals the probability that the mortgage survives to month $t - 1$ multiplied by the monthly probability of prepayment: $\hat{\phi}(1 - \hat{\phi})^{t-1}$. Finally, the cash flow in month t associated with the full prepayment of the mortgage in an earlier month is zero. The probability of this outcome is one minus the probability that the loan survives to month $t - 1$: $1 - (1 - \hat{\phi})^{t-1}$. The expectation of the cash flow at month t is given by the following expression:

$$E(C_t) = \hat{A}\,(1 - \hat{\phi})^t + \hat{B}_t\hat{\phi}\,(1 - \hat{\phi})^{t-1}$$

which has the same form as equation (6).

Following Navratil (1985) and others, the monthly prepayment rate is assumed to vary inversely with the difference between the current rate and the contract rate.[7] Thus, the expectation in equation (6) is conditional on the difference between the current rate i and the contract rate i_0.

The conditional expectation of the cash flow from the MBS in month t may be written as

$$E(C_t|i) = A(1 - \phi)^{t-1} + B_t\phi(1 - \phi)^{t-1} \tag{7}$$

The duration measure $\mathbb{D}$ for an MBS defined with respect to the current rate is[8]

$$\mathbb{D} = -\frac{\partial M_0}{\partial i} \cdot \frac{(1 + i)}{M_0} \tag{8}$$

Applying the definition of duration to equation (1), we obtain

[7]Although ϕ is modeled here as a constant across time t, it is not assumed to be constant for different mortgage pools. In general, ϕ depends upon the geographical location of most or all of the property used to secure the mortgages and demographic makeup of a pool.

[8]An alternative measure of duration that does not involve a direct calculation is called "empirical duration." Under this method, duration is estimated from the historical relationship between prices and yields (see Pinkus and Chandoha (1987)).

$$\mathbb{D} = -\left\{\sum_{t=1}^{T}\left[\frac{\partial E(C_t|i)}{\partial i}\cdot\frac{1}{(1+y)^t} - \frac{\partial y}{\partial i}\cdot\frac{tE(C_t|i)}{(1+y)^{t+1}}\right]\right\}\cdot\frac{(1+i)}{M_0} \tag{9}$$

Differentiating equation (7) with respect to i yields the change in conditional expectation with respect to a change in the current rate:

$$\frac{\partial E(C_t|i)}{\partial i} = \frac{\partial \phi}{\partial i}\cdot\frac{1}{(1-\phi)}\cdot\left[-tE(C_t|i) + B_t\,(1-\phi)^{t-1}\right] \tag{10}$$

Substituting the result from equation (10) into (9) yields:

$$\mathbb{D} = \left\{\frac{(1+i)}{M_0}\cdot\frac{\partial \phi}{\partial i}\cdot\frac{1}{(1-\phi)}\sum_{t=1}^{T}\left[\frac{tE(C_t|i)}{(1+y)^t} - \frac{B_t\,(1-\phi)^{t-1}}{(1+y)^t}\right]\right.$$
$$\left. + \frac{\partial y}{\partial i}\cdot\frac{(1+i)}{(1+y)}\cdot\frac{1}{M_0}\cdot\sum_{t=1}^{T}\frac{tE(C_t|i)}{(1+y)^t}\right\} \tag{11}$$

Note that the definition of the conventional duration[9]

$$D^M = \frac{1}{M_0}\sum_{t=1}^{T}\frac{tE(C_t|i)}{(1+y)^t} \tag{12}$$

occurs twice in equation (11). Substituting equation (12) into equation (11) yields

$$\mathbb{D} = \varepsilon_y D^M - \varepsilon_s\left[D^M - \frac{1}{\phi M_0}\cdot\sum_{t=1}^{T}\frac{B_t\,(1-\phi)^t}{(1+y)^t}\right] \tag{13}$$

where ε_y is the elasticity of the cash-flow yield with respect to the current rate (hereafter called the yield elasticity):

$$\varepsilon_y = \frac{\partial y}{\partial i}\cdot\frac{(1+i)}{(1+y)}$$

and where ε_s is the elasticity of the monthly survival rate $(1-\phi)$ with respect to the current rate (hereafter called the survival elasticity):

[9]This definition of duration is also referred to in the literature as Macaulay's duration. Note that when the expected cash flow depends upon the interest rate, Macaulay's definition does not fully capture the price sensitivity of the mortgage.

$$\varepsilon_s = -\frac{\partial \phi}{\partial i} \cdot \frac{(1 + i)}{(1 - \phi)}$$

The conventional duration is the correct measure for an MBS only if the yield elasticity equals one and the second term in equation (13) equals zero. The second term equals zero under two conditions: (1) the survival elasticity equals zero or (2) the expression in the brackets equals zero. If the survival elasticity equals zero, then by definition the expected cash flows do not vary with the current rate and the conventional duration applies. The expression in the brackets equals zero if the current rate equals the contract rate.[10] Since these conditions do not hold in general, the conventional duration measure is biased. The extent of the bias increases with the magnitude of the prepayment elasticity and with the magnitude of the deviation of the current rate from the contract rate.

III. An Example of Alternative Duration Measures

In this section, three MBS duration measures are examined: (1) constant prepayment duration (CP-Dur), in which the prepayment rate is assumed to be constant; (2) variable prepayment duration (VP-Dur), defined in equation (12); and (3) elastic prepayment duration (EP-Dur), developed in this paper and defined in equation (13). These three measures are evaluated for a thirty-year mortgage bond with a remaining life of twenty-five years making monthly payments at a contract rate of 12 percent compounded monthly.

Throughout the illustration, Navratil's (1985) estimates of the prepayment rate function are used in computing durations.[11] Prepayment rates and elasticities for a 12 percent mortgage pool at different current rates are reported in Table 1.[12]

The elasticities reported in Table 1 are positive because the monthly survival rate is positively related to the difference between the current rate and the contract rate. The relation is especially strong if the current rate is less than the

[10]Proof of this statement is available from the authors on request. The proof depends upon the fact that when the current rate equals the contract rate, the present value of the mortgage is independent of the prepayment rate.

[11]Point estimates for annual prepayment rates for the 12 percent mortgage pools are computed from Navratil's logistic function. Point elasticities are calculated by applying the definition of the survival elasticity to the same logistic function.

[12]Elasticities presented in Table 1 are defined differently from those in Table 3 in Navratil's paper. The values reported in his Table 3 are arc-elasticities of the prepayment rate with respect to the current rate. The elasticities in Table 1 of this paper are point elasticities of the survival rate with respect to the current rate.

TABLE 1. Prepayment Elasticities and Alternative Duration Measures.

Current Rate Differential[b]	Survival Elasticity	Prepayment Rate	Duration[a]		
			CP-Dur	VP-Dur	EP-Dur
(200)	7.320	8.400%	7.155	5.018	1.948
(175)	6.111	6.996%	7.068	5.299	2.804
(150)	5.088	5.812%	6.982	5.550	3.598
(125)	4.227	4.818%	6.898	5.767	4.306
(100)	3.506	3.987%	6.815	5.947	4.912
(75)	2.903	3.294%	6.733	6.101	5.421
(50)	2.401	2.719%	6.653	6.209	5.818
(25)	1.984	2.241%	6.574	6.308	6.140
0	1.340	1.510%	6.496	6.496	6.496
25	0.373	1.426%	6.419	6.446	6.479
50	0.353	1.346%	6.343	6.397	6.460
75	0.334	1.271%	6.269	6.350	6.440
100	0.316	1.200%	6.196	6.302	6.413
125	0.299	1.133%	6.124	6.249	6.383
150	0.283	1.069%	6.053	6.188	6.339
175	0.268	1.009%	5.983	6.146	6.314
200	0.253	0.953%	5.914	6.078	6.258

[a]CP-Dur is the constant prepayment duration in which the prepayment rate is assumed to be constant; VP-Dur is the variable prepayment duration defined in equation (12); and EP-Dur is the elastic prepayment duration developed in this paper, defined in equation (13).
[b]The contract rate is 12 percent and the current rate differential $(i - i_0)$ is measured in basis points.

contract rate. On the other hand, if the current rate is greater than the contract rate, the elasticities are fairly constant.

The CP-Dur is computed based on the assumption of a constant annualized prepayment rate of 1.51 percent for all current rates.[13] The VP-Dur is computed according to equation (12) with expected cash flows for a given current rate determined by the CPR in Table 1. The EP-Dur is computed from equation (13) with the CPR and the survival elasticity reported in Table 1.

Figure I presents the three duration measures versus the difference between the current rate and contract rate. For a positive current rate differential, several observations can be made. First, the differences among the three duration measures are small because the survival elasticity associated with a positive current rate differential is small. For example, at a current rate differential of positive 200 basis points, the three values for the CP-Dur, the VP-Dur, and the EP-Dur are 5.914, 6.078, and 6.258, respectively. The variation from the highest to the lowest duration value is less than 6 percent of the EP-Dur value. Second, the EP-Dur exceeds the VP-Dur, which exceeds the CP-Dur. Third, all three durations are declining functions of the difference between the current rate and contract rate. Finally, since the prepayment elasticity associated with a

[13]The CPR of 1.51 percent is the estimated prepayment rate for a 12 percent pool when the current rate equals the contract rate (from the third column of Table 1).

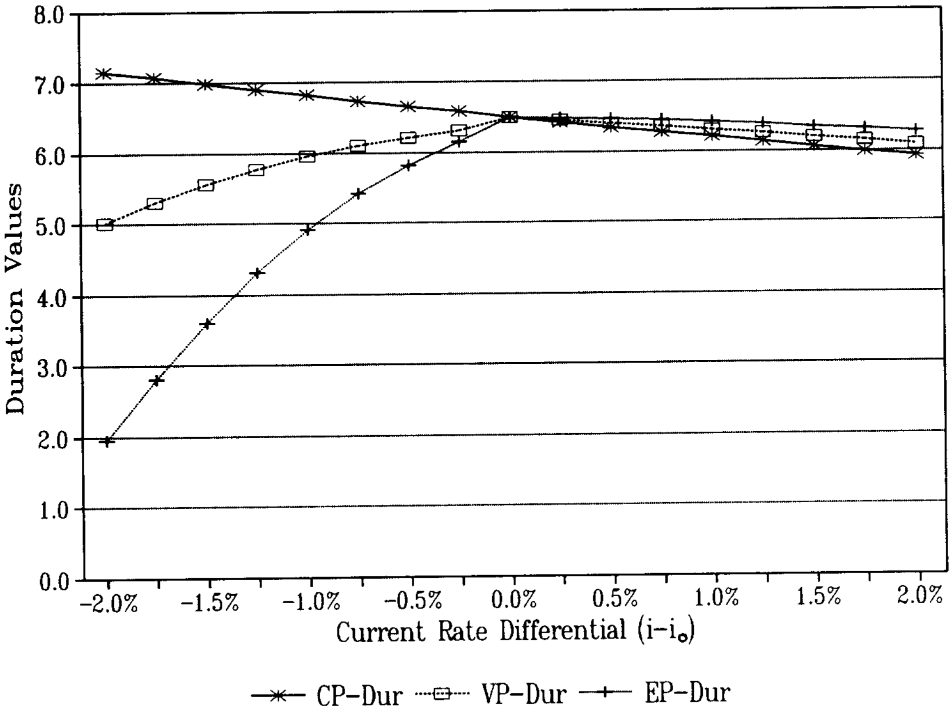

Figure I. Comparison of Mortgage-Backed Security Duration Measures at Varying Current Rate Differentials.

positive current rate differential is relatively insensitive to a change in the current rate differential, all three duration measures are relatively insensitive to changes in the current rate. A change in the current rate from 0 to 200 basis points above the contract rate results in a decline of less than 10 percent for each of the three duration measures.

For a negative current rate differential, the change in the survival elasticity for a given change in the current rate is much greater than for a positive current rate differential. Therefore, large variations among the three durations result when the current rate falls below the contract rate. For a current rate differential of minus 200 basis points, the values for the CP-Dur, the VP-Dur, and the EP-Dur are 7.155, 5.018, and 1.948, respectively. The CP-Dur is over 140 percent of the VP-Dur estimate and over 350 percent of the EP-Dur estimate. This contrasts sharply with the modest 6 percent variation among the three duration measures associated with a positive 200-basis-point current rate differential. Also, in contrast to results for the positive current differential, the duration curves are not all downward sloping. The CP-Dur curve is downward sloping over the entire range of current rate differentials, but both the VP-Dur and the EP-Dur curves are upward sloping when the current rate is less than the contract rate and downward sloping when the current rate is greater than the contract rate.

The slope of the duration curve for a given level of interest rate is commonly referred to as convexity. Like the duration of a fixed-income security, the CP-Dur is convex. However, the VP-Dur and EP-Dur measures demonstrate a negative convexity for a negative current rate differential. Convexity is important in implementing immunization strategies. Strictly speaking, for instruments with negative convexity, immunization through a duration-matching strategy is not feasible unless the matching liabilities also exhibit negative convexity (Jacob and Toevs (1988)). The negative convexity of the VP-Dur and the EP-Dur models places additional constraints on the traditional duration-matching immunization approaches to risk control when managing MBS securities.

IV. Conclusion

In this paper a duration measure is developed for mortgage-backed pass-through securities that incorporates the sensitivity of the prepayment rate to changes in interest rates. Conventional duration measures for mortgage-backed pass-through securities, which fail to account for this dependence between prepayment and the current rate, overstate the duration on a bond whose current market rate is less than the contract rate.

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