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Positively Weighted Minimum-Variance Portfolios and the Structure of Asset Expected Returns

Michael J. Best and Robert R. Grauer*

Abstract
In this paper, we derive simple, directly computable conditions for minimum-variance portfolios to have all positive weights. We show that either there is no minimum-variance portfolio with all positive weights or there is a single segment of the minimum-variance frontier for which all portfolios have positive weights. Then, we examine the likelihood of observing positively weighted minimum-variance portfolios. Analytical and computational results suggest that: i) even if the mean vector and covariance matrix are compatible with a given positively weighted portfolio being mean-variance efficient, the proportion of the minimum-variance frontier containing positively weighted portfolios is small and decreases as the number of assets in the universe increases, and ii) small perturbations in the means will likely lead to no positively weighted minimum-variance portfolios.

I. Introduction

Minimum-variance and positively weighted portfolios play central roles in models of asset pricing and in portfolio theory.1 The two sets of portfolios tend to go hand in hand in many models of asset pricing, particularly in the Capital Asset Pricing Model (CAPM) of Sharpe (1964) andLintner (1965), that predicts the positively weighted market portfolio will be mean-variance (MV) efficient. Yet, positively weighted portfolios are hardly ever observed when minimum-variance portfolios are generated from historical data.2 In light of this apparent contradiction, it would be interesting to know the conditions under which the minimum-

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1By definition, a minimum-variance portfolio minimizes the variance of return for a given level of expected return. Also, by definition, a minimum-variance portfolio whose expected return exceeds the expected return on the global minimum-variance portfolio is mean-variance efficient.

2See the references in Best and Grauer (1985) and Green (1986).
variance frontier contains portfolios with all positive weights.\textsuperscript{3} For example, it would be interesting to know whether short-sales constraints are likely to be binding, or whether mutual or pension fund portfolios, that are characteristically all positively weighted, are likely to be MV efficient. Perhaps more importantly, it would be interesting to know how robust our models of asset pricing are to small perturbations in the parameters. For example, it would be interesting to know whether there are any positively weighted minimum-variance portfolios if asset means do not plot exactly on the Security Market Line (SML).

Roll (1977), Rudd (1977), Roll and Ross (1977), and Green (1986) examined the conditions under which the minimum-variance frontier contains positively weighted portfolios. With the exception of Rudd, they attempted to provide "qualitative" conditions for minimum-variance portfolios to have all positive weights without inverting the covariance matrix. Green's results are the most general. He employed duality theory to provide necessary and sufficient conditions for the nonexistence of positively weighted minimum-variance portfolios. That is, he showed that there are no positively weighted minimum-variance portfolios if and only if a portfolio exists such that its weights sum to zero, its mean is zero, and it is nonnegatively correlated with all assets. However, as a practical matter, it is necessary to invert the covariance matrix in order to find the portfolio. Therefore, in this paper, we focus on the primal problem.

Section II contains the theory. First, given any mean vector and a positive-definite covariance matrix, we derive simple, directly computable conditions for the resulting minimum-variance portfolios to have all positive weights. We show either there are no positively weighted minimum-variance portfolios or there is a single segment of the minimum-variance frontier for which all portfolios have positive weights. (In the Appendix, we extend the analysis to the case where the weights of minimum-variance portfolios lie within upper and lower bounds.) Then, we investigate the likelihood of observing positively weighted minimum-variance portfolios. We begin by assuming that there is at least one positively weighted minimum-variance portfolio. Call this portfolio $x^*$, where $x^*$ is an $n$-vector of positive portfolio weights. In an $n$-asset universe, the portfolio $x^*$ is MV efficient if and only if the mean vector is $(\Sigma, x^*)$-compatible (see Best and Grauer (1985)), i.e., if and only if

$$\mu^* = \theta_1 \underline{\mu} + \theta_2 \Sigma x^*,$$

where $\mu^*$ and $\underline{\mu}$ are $n$-vectors containing one plus the expected rates of return on the $n$-assets and ones, respectively; $\Sigma$ is an $(n, n)$ positive-definite covariance matrix of asset returns; $\theta_1$ and $\theta_2$ are (positive) scalar constants; and the $i$th element of the vector $\Sigma x^*$ is the covariance of the return on asset $i$ with the return on the portfolio $x^*$. A special case has played a central role in the financial theory: if $x_m$ is the market portfolio, then the CAPM predicts that prices will adjust until the means are $(\Sigma, x_m)$-compatible. In that case, (1) becomes the SML. Depending on how we view (1), there are two ways of interpreting the results in this paper. The first suggests that positively weighted minimum-variance portfolios are unlikely to

\textsuperscript{3}The expressions, \textit{positively weighted portfolios}, \textit{nonnegative portfolios}, and \textit{portfolios with positive weights}, are taken to be synonymous with the expression, \textit{portfolios with all positive weights}. 
exist. On the other hand, if we believe that positively weighted minimum-variance portfolios arise as a consequence of some equilibrium condition, then they must exist. In that case, our results indicate that the equilibrium conditions of the CAPM impose quite a rigid structure on asset expected returns.\footnote{We thank the referee for suggesting this second interpretation. As the reader can see, the point is debatable.}

To be more specific, we explore two questions in the \((\Sigma, \bar{x}^*)\)-compatible means framework: 1) What happens to the set of positively weighted minimum-variance portfolios as the number of assets in the universe increases? and 2) What happens to the set of positively weighted minimum-variance portfolios if the means are not exactly \((\Sigma, \bar{x}^*)\)-compatible? First, we show that, under reasonable conditions, as the number of assets in the universe increases, the segment of the frontier containing positively weighted portfolios converges to a single point, i.e., \(\bar{x}^*\) may be the \textit{only} positively weighted minimum-variance portfolio. Second, we draw on results developed in Best and Grauer (1991a), (1991b), (1992) which suggest that the composition of MV-efficient portfolios may be extremely sensitive to changes in the means. In other words, small perturbations in the \((\Sigma, \bar{x}^*)\)-compatible means will likely mean there are no positively weighted minimum-variance portfolios.

The analysis highlights the importance of both the mean vector and covariance matrix in determining whether there are positively weighted minimum-variance portfolios. In two senses, the means play the central role. First, the \((\Sigma, \bar{x}^*)\)-compatible means framework makes it clear that one cannot rule out the possibility of positively weighted minimum-variance portfolios based on an analysis of either the covariance matrix or the global minimum-variance portfolio alone. The means can always be chosen to make any positively weighted portfolio MV efficient.\footnote{In addition, notice that the CAPM simply predicts that prices will adjust so that the means are \((\Sigma, \bar{x}_m)\)-compatible. It predicts nothing about the composition of the global minimum-variance portfolio.}

Second, the composition of minimum-variance portfolios is extremely sensitive to perturbations in the means. But, in another sense, the covariance matrix plays the key role as it determines the sensitivity of the portfolio weights to changes in the means.

In Sections III and IV, we employ the conditions developed in Section II to examine whether there are positively weighted minimum-variance portfolios when covariance matrices are generated from return data drawn from the Center for Research in Security Prices (CRSP) data base. It is well documented that, with historical inputs, positively weighted MV-efficient portfolios are the exception rather than the rule, and it is tempting to attribute the lack of positively weighted portfolios to measurement error. However, the extreme sensitivity of the portfolio weights to changes in the asset means we document suggests that this explanation is too simple.\footnote{See Bawa, Brown, and Klein (1979) for an excellent discussion of the estimation risk problem.} That is, while the magnitude of the errors induced by sampling swamps the magnitude of the perturbations in the \((\Sigma, \bar{x}^*)\)-compatible means, the perturbations themselves lead to a lack of positively weighted minimum-variance portfolios.\footnote{A second interpretation has been offered by the referee. "My feeling is that this paper’s results demonstrate that measurement error (which I would take to mean the lack of precision with which historical returns allow us to estimate the means) is exactly the reason we seldom see positive efficient...} For asset universes ranging from 10 to 100 assets, we use historical...
data to generate covariance matrices and construct means to make a positively weighted portfolio MV efficient. Then, the computational analysis substantiates the theoretical analysis: i) the portion of the minimum-variance frontier containing positively weighted portfolios is relatively small and decreases as the number of assets increases, and ii) small perturbations of the \((\Sigma, \lambda^0)\)-compatible mean vector results in there being no positively weighted minimum-variance portfolios. Clearly, (1) imposes quite a rigid structure on asset expected returns. In Section V, we outline some of the implications this structure has for portfolio management, for tests of the MV efficiency of a portfolio, for forecasting expected returns, for using the SML to measure investment performance, and for models of asset pricing.

II. The Theory

We begin with a short statement of the efficient set mathematics as it provides the basis for the subsequent analysis. Consider a universe of \(n\)-risky assets. Let \(\Sigma\) denote their \((n, n)\)-covariance matrix, \(\mu\) the \(n\)-vector of their expected returns, and let \(\lambda\) denote the \(n\)-vector whose components are all unity. Assume that \(\Sigma\) is positive definite. The MV problem (see Markowitz (1970), Sharpe (1970), or Best and Grauer (1990)) is

\[
(2) \quad \max \left\{ t \mu' \bar{x} - 1/2 \bar{x}' \Sigma \bar{x} \mid \lambda' \bar{x} = 1 \right\},
\]

where \(t\) is a scalar parameter, \(\bar{x}\) is an \(n\)-vector of portfolio weights, and \(\lambda' \bar{x} = 1\) is the budget constraint. There are two closely related ways of interpreting (2). First, (2) may be interpreted as a parametric quadratic programming (PQP) problem where the minimum-variance frontier is traced out as the parameter \(t\) varies from \(-\infty\) to \(\infty\). This interpretation is used to develop a direct characterization of positively weighted minimum-variance portfolios. On the other hand, positive values of \(t\) are of primary economic interest as they yield MV-efficient portfolios. In this case, it is convenient to think of \(t\) as an MV investor’s risk tolerance parameter, where the larger \(t\) is, the more tolerant the investor is to risk. When \(t\) is fixed at some value, say \(t = T\), the solution to (2) yields the MV-efficient portfolio for the investor with risk tolerance parameter \(T\). This provides an important economic interpretation of the MV problem as well as a convenient way of motivating an examination of how changes in the means affect the investor’s optimal portfolio.

The first-order conditions for (2) are

\[
(3) \quad \Sigma \bar{x} + \lambda \lambda = t \mu, \quad \text{and} \quad \lambda' \bar{x} = 1,
\]

portfolios based on historical data. This paper demonstrates that our estimates of the means would have to be quite accurate before they would give even the market portfolio as an efficient portfolio. The fact that we always perform our empirical studies on a subset of the investment universe probably makes the problem worse.” Again, we have agreed to disagree.

\(8\) Minimum-variance portfolios may be defined in other equivalent ways. One such way, popularized by Merton (1972), and Roll (1977) is

\[
\min \left\{ 1/2 \lambda' \Sigma \lambda - 1, \mu' \bar{x} = \mu_p \right\}.
\]

The minimum-variance frontier is traced out as \(\mu_p\) is varied, and the multiplier associated with the constraint \(\mu' \bar{x} = \mu_p\) corresponds to the risk tolerance parameter \(t\) in (2). Either formulation may be used. However, in what follows, it is more convenient to employ (2).
where $\lambda$ is the multiplier for the budget constraint. Solving these equations, the optimal portfolio and multiplier are

$$x(t) = \Sigma^{-1} \mu/c + t \left[ \Sigma^{-1} \left( \mu - \zeta a/c \right) \right],$$

and

$$\lambda(t) = -1/c + t a/c,$$

where efficient set constants $a$, $b$, and $c$ are defined as

$$a = \zeta' \Sigma^{-1} \mu, \quad b = \mu' \Sigma^{-1} \mu, \quad \text{and} \quad c = \zeta' \Sigma^{-1} \zeta.$$

It is useful to express $x(t)$ as

$$x(t) = h_0 + t h_1,$$

where

$$h_0 = \Sigma^{-1} \mu/c, \quad \text{and} \quad h_1 = \Sigma^{-1} \left( \mu - \zeta a/c \right).$$

Note that $h_0$ is the global minimum-variance portfolio. Furthermore, observe that $h_1 = 0$ if and only if $\mu$ is a multiple of $\zeta$. Since this case is trivial, i.e., all the means are equal, we assume it away. By the definition of $h_0$, $\zeta' h_0 = 0$. Since $h_1 \neq 0$ and its components sum to zero, $h_1$ must have at least one component that is strictly positive and at least one component that is strictly negative. This observation simplifies the subsequent analysis.

The expected return, $\mu_p = \mu' x$, and variance, $\sigma^2_p = x' \Sigma x$, of the optimal portfolio are linear and quadratic functions of $t$,

$$\mu_p = \alpha_0 + \alpha_1 t, \quad \text{and} \quad \sigma^2_p = \gamma_0 + \gamma_1 t + \gamma_2 t^2,$$

where $\alpha_0 = \mu' h_0 = a/c$, $\alpha_1 = \mu' h_1 = b - a^2/c$, $\gamma_0 = h_0' \Sigma h_0 = 1/c$, $\gamma_1 = 2 h_0' \Sigma h_1 = 0$, and $\gamma_2 = h_1' \Sigma h_1 = \alpha_1$. Eliminating $t$ from (8), we obtain the equation of the minimum-variance frontier,

$$(\mu_p - \alpha_0)^2 = \alpha_1 \left( \sigma^2_p - \gamma_0 \right),$$

which is a parabola (hyperbola) in MV (mean-standard deviation) space.

A. A Direct Characterization of Positively Weighted Efficient Portfolios

Equation (6) shows that each component of $x(t)$ is a linear function of $t$, i.e., $x_i(t) = h_{0i} + t h_{1i}$, $i = 1, \ldots, n$. If $h_{1i} > 0$, then $x_i(t)$ is increasing in $t$ and will be nonnegative provided that $t \geq -h_{0i}/h_{1i}$. Let

$$t_{\ell} = \max \left\{ -h_{0i}/h_{1i} \mid \text{all } i \text{ with } h_{1i} > 0 \right\},$$

and

$$t_u = \min \left\{ -h_{0i}/h_{1i} \mid \text{all } i \text{ with } h_{1i} < 0 \right\}.$$
It follows that

\[
x_i(t) \geq 0 \text{ for all } i \text{ with } h_{1i} > 0 \text{ and for all } t \geq t_\ell,
\]

\[
x_i(t) \geq 0 \text{ for all } i \text{ with } h_{1i} < 0 \text{ and for all } t \leq t_u.
\]

It is possible, although perhaps unlikely, to have \( h_{1i} = 0 \) for one or more assets \( i \). In this case, \( x_i(t) = h_{0i} \) remains fixed for all \( t \). In order for \( x(t) \) to be nonnegative for some \( t \), it is necessary that

\[
h_{0i} \geq 0 \text{ for all } i \text{ such that } h_{1i} = 0.
\]

The above discussion implies that \( x(t) \geq 0 \) if and only if \( t_\ell \leq t \leq t_u \) and (12) is satisfied. Two conclusions follow from this. First, the assertion that no minimum-variance portfolio has nonnegative weights is equivalent to the condition that either \( t_\ell > t_u \) or there exists a \( k \) such that \( h_{1k} = 0 \) and \( h_{0k} < 0 \). Second, if \( t_\ell \leq t_u \) and (12) is satisfied, then all minimum-variance portfolios \( x(t) \) are nonnegative for all \( t \) satisfying \( t_\ell \leq t \leq t_u \). Thus, there are either no nonnegative minimum-variance portfolios or there are some and they all correspond to the interval \( t_\ell \leq t \leq t_u \). These two situations are illustrated (for a four-asset universe) in Figures 1 and 2, respectively.

Define \( \mu_p = \mu' x(t_\ell) = \mu' h_0 + t_\ell \mu' h_1 \), and \( \mu_{pu} = \mu' x(t_u) = \mu' h_0 + t_u \mu' h_1 \). Since \( \mu_p(t) \) is monotonic increasing in \( t \), we can reformulate our previous result.
as follows. The assertion that no minimum-variance portfolio has nonnegative weights is equivalent to the condition that \( \mu_{p_\ell} > \mu_{p_u} \) or that (12) is not satisfied. Furthermore, if \( \mu_{p_\ell} \leq \mu_{p_u} \) and (12) is satisfied, then all portfolios corresponding to the segment of the minimum-variance frontier defined by \( \mu_{p_\ell} \leq \mu_p \leq \mu_{p_u} \) have nonnegative weights.\(^9\) This is illustrated in Figure 3.

Summarizing the analysis we have the following theorem.

**Theorem 1.**

(a) There are nonnegatively weighted minimum-variance portfolios if and only if \( t_\ell \leq t_u \), and \( h_{3i} \geq 0 \) for all \( i \) with \( h_{4i} = 0 \). The nonnegatively weighted minimum-variance portfolios are given by \( \bar{x}(t) = b_0 + t h_1 \) for all \( t \) satisfying \( t_\ell \leq t \leq t_u \) and correspond to the segment of the minimum-variance frontier defined by \( \mu_{p_\ell} \leq \mu_p \leq \mu_{p_u} \).

(b) A necessary and sufficient condition that some minimum-variance portfolio, \( \bar{x}(t) \), not have all strictly positive weights is that either (i) \( t < t_\ell \) or \( t > t_u \), or (ii) \( h_{1k} = 0 \) for some \( k \) with \( h_{0k} < 0 \), or both.

(c) A necessary and sufficient condition that no minimum-variance portfolio have nonnegative weights is either \( t_\ell > t_u \), or \( h_{0k} = 0 \) for some \( k \) with \( h_{1k} < 0 \), or both.\(^{10}\)

\(^9\)Similarly, defining

\[
\sigma_{p_\ell}^2 = b_0' \Sigma b_0 + t_\ell^2 h_1' \Sigma h_1, \quad \text{and} \quad \sigma_{p_u}^2 = b_0' \Sigma b_0 + t_u^2 h_1' \Sigma h_1,
\]

an equivalent result can be formulated in terms of \( \sigma_{p_\ell}^2 \) and \( \sigma_{p_u}^2 \).

\(^{10}\)Parts (b) and (c) of our theorem must be equivalent to Green’s Theorems 1 and 2, respectively. An appendix showing the detailed relationship between the two characterizations is available on request.
The argument is noteworthy for its simplicity. To paraphrase the analysis, an immediate and obvious consequence of the efficient set mathematics is that the weights in minimum-variance portfolios, as well as their means, are linear functions of a parameter \( t \). As \( t \) increases, some of the asset weights increase and some decrease.\(^{11}\) There are two possibilities. First, if the weight of the last increasing weight asset turns positive (at \( t = \ell \)) after the weight of the first decreasing weight asset turns negative (at \( t = u \)), then there are no positively weighted minimum-variance portfolios. (That is, there are no positively weighted minimum-variance portfolios if and only if \( \ell > u \).) Second, if the weight of the last increasing weight asset turns positive (at \( t = \ell \)) before the weight of the first decreasing weight asset turns negative (at \( t = u \)), then there are positively weighted minimum-variance portfolios in the range \( \ell \leq t \leq u \). Furthermore, these portfolios lie on a single segment of the minimum-variance frontier.

While the argument is simple, it is powerful. In the Appendix, we extend it to the case where the weights of minimum-variance portfolios lie between upper and lower bounds. In the next section, we extend it to show what happens to the segment of the frontier containing positively weighted portfolios as the number of assets in the universe increases.

\(^{11}\)To keep things as simple as possible, assume no weight is constant in \( t \).
B. The Effect of Increasing the Number of Assets in the Universe

Suppose that the frontier contains at least one positively weighted portfolio and we want to know what happens to the set of positively weighted minimum-variance portfolios as well as the segment of the minimum-variance frontier that contains them, as the number of assets in the universe increases. As noted in the Introduction, given any positive-definite covariance matrix, the positively weighted portfolio \( \chi^* \) is MV efficient if and only if (1) holds.\(^{12}\) While there are an infinite number of \((\Sigma, \chi^*)\)-compatible means from the two-parameter family in (1), the values of \( \theta_1 \) and \( \theta_2 \) are constrained by the following economics. The intercept of (1) must equal one plus the zero-beta rate, \( r_z \), which, in turn, is equal to the multiplier divided by the risk tolerance parameter. Furthermore, the slope must equal the expected excess return of the optimal portfolio divided by its variance, as well as the reciprocal of investor risk tolerance. That is,

\[
\theta_1 = r_z = \lambda^*/t^*, \quad \text{and} \quad \theta_2 = (\mu_p - r_z)/\sigma_p^2 = 1/t^*.
\]

The \((\Sigma, \chi^*)\)-compatible means allow us to write \( h_1 \) as \( \theta_2 \) times the difference between \( \chi^* \) and the global minimum-variance portfolio \( h_0 \). That is, from (6),

\[
\chi^* = h_0 + t^* h_1.
\]

Therefore,

\[
h_1 = 1/t^* [\chi^* - h_0] = \theta_2 [\chi^* - h_0].
\]

Then, we have the following theorem.

**Theorem 2.**

Consider the mean-variance problem (2). Let \( \chi^* \) be any positively weighted portfolio. Assume that \( \Sigma \) is positive definite and that \( \mu \) is chosen so that \( \chi^* \) is mean-variance efficient. Let \( t^* \) be its associated parameter and \( h_0 \) be the global minimum-variance portfolio. Finally, let

\[
\delta_\ell(n) = \max \{ x_i^*/h_0 \mid \text{all } i \text{ with } x_i^* > h_0 \},
\]

and

\[
\delta_u(n) = \min \{ x_i^*/h_0 \mid \text{all } i \text{ with } x_i^* < h_0 \},
\]

where \( n \) is the number of assets. Then:

(a) As \( n \to \infty \), the segment of the minimum-variance frontier containing positively weighted portfolios converges to the single point \( \chi^* \) if and only if

\[
\lim_{n \to \infty} \delta_\ell(n) = \lim_{n \to \infty} \delta_u(n) = 0.
\]

(b) If the global minimum-variance portfolio is positively weighted, then as \( n \to \infty \), the positively weighted portion of the minimum-variance frontier converges to the segment whose end points correspond to \( h_0 \) and \( \chi^* \) if and only if

\[
\lim_{n \to \infty} \delta_\ell(n) = \infty \quad \text{and} \quad \lim_{n \to \infty} \delta_u(n) = 0.
\]

\(^{12}\) Equivalently, we could write (1) as \( \mu^* = \theta_1 \ell + \theta_3 \beta^* \), where \( \beta^* = (\Sigma \chi^*)/(\chi^* \Sigma \chi^*) \), and \( \theta_3 = \theta_2 \ell^*/\Sigma \chi^* \).
Proof Part (a). From (10), (11), and (14), the expressions for \( t_{\ell} \) and \( t_u \) become

\[
 t_{\ell} = t^* \max \left\{ \frac{h_{0i}}{[h_{0i} - x_i]} \right\} \quad \text{all } i \text{ with } x_i^* > h_{0i} \n\]

and

\[
 t_u = t^* \min \left\{ \frac{h_{0i}}{[h_{0i} - x_i]} \right\} \quad \text{all } i \text{ with } x_i^* < h_{0i} \n\]

Dividing the top and bottom of the maximand and minimand by \( h_{0i} \) gives the equivalent expressions for \( t_{\ell} \) and \( t_u \),

\[
 t_{\ell} = t^* \max \left\{ \frac{1}{(1 - x_i^*/h_{0i})} \right\} \quad \text{all } i \text{ with } x_i^* > h_{0i} \n\]

\[
 t_u = t^* \min \left\{ \frac{1}{(1 - x_i^*/h_{0i})} \right\} \quad \text{all } i \text{ with } x_i^* < h_{0i} \n\]

It is straightforward to verify that

\[
 t_{\ell} = t^* \left( 1 / (1 - \delta_{\ell}(n)) \right), \quad \text{and} \quad t_u = t^* \left( 1 / (1 - \delta_u(n)) \right),
\]

where \( \delta_{\ell}(n) \) and \( \delta_u(n) \) are as in the statement of the theorem. For any function \( f(n) \), it follows that

\[
 \lim_{n \to \infty} 1 / (1 - f(n)) = 1 \quad \text{if and only if} \quad \lim_{n \to \infty} f(n) = 0.
\]

From this last observation and (15), we have

\[
 \lim_{n \to \infty} t_{\ell} = t^* \quad \text{if and only if} \quad \lim_{n \to \infty} \delta_{\ell}(n) = 0,
\]

and

\[
 \lim_{n \to \infty} t_u = t^* \quad \text{if and only if} \quad \lim_{n \to \infty} \delta_u(n) = 0,
\]

from which part (a) of the theorem follows.

Proof Part (b). The proof of part (b) is similar to part (a) with the observation from (15) that

\[
 \lim_{n \to \infty} t_{\ell} = 0 \quad \text{if and only if} \quad \lim_{n \to \infty} \delta_{\ell}(n) = \infty. \quad \Box
\]

Special cases provide intuition. In the first case, assume that \( x^* \) is equally weighted. Then, it follows from (10) and (11) that the assets which determine \( t_{\ell} \) and \( t_u \) are the ones whose weights in the global minimum-variance portfolio are the smallest (\( h_{0\text{min}} \)) and largest (\( h_{0\text{max}} \)), respectively. Part (a) of the theorem asserts that the segment of the frontier containing positively weighted portfolios converges to a single point \( x^* \) if and only if the ratios \((1/n)/h_{0\text{min}}\) and \((1/n)/h_{0\text{max}}\) converge to zero. In the computational sections, we show that when the equally weighted portfolio is MV efficient in 10 to 100 asset universes, the segment of the frontier containing positively weighted portfolios shrinks as \( n \) increases. This
follows from the fact that $h_{0\text{min}}$ and $h_{0\text{max}}$ are $-0.042$ and $0.441$ in the 10-asset universe and $-0.191$ and $0.268$ in the 100-asset universe.

More generally, if the market values of U.S. equities are representative, a market-value weighted portfolio of $n$ assets will contain weights that are both large and small relative to $1/n$. (But there is no reason to expect the corresponding elements of $h_0$ will be either large or small.) In this case, $t_t$ and $t_u$ will most likely be determined from the assets whose weights in $x^*$ are small, and whose corresponding weights in $h_0$ are either large and negative or large and positive, respectively. Now, if some of the weights in $x^*$ are small relative to $1/n$, intuition would suggest that the proportion of the frontier containing positive weights will be small relative to the equally weighted case. In fact, in the computational sections, we consider two sum-of-the-digits weighting schemes for $x^*$ and, in the 100-asset universe, find that the proportion of the frontier containing positively weighted portfolios is on the order of one quarter to one tenth the size of the equally weighted case. Finally, assume that the covariance matrix is diagonal—an unreasonable assumption for stock market data. In this case, the weights in the global minimum-variance portfolio are positive and part (b) of the theorem shows when the positively weighted portion of the frontier will shrink to a segment containing $h_0$ and $x^*$.

C. The Effect of Perturbations in the $(\Sigma, x^*)$-Compatible Means

Now, suppose we refocus and ask what happens to the composition of a minimum-variance portfolio if the means are not exactly $(\Sigma, x^*)$-compatible. It is both convenient and informative to cast the problem in terms of a specific investor with risk tolerance parameter $T$. In the context of the problem at hand, we may wish to be even more specific and assume that $T = t^*$ so that the investor holds the portfolio $x^*$. Therefore, assume that (2) has been solved for a specific investor with risk tolerance parameter $T$ and suppose the investor wishes to know how the optimal solution for (2) depends on $\mu$. The analysis may be performed by solving the following closely related PQP problem,

\begin{equation}
(16) \quad \max \left\{ T \left( \mu + tq \right)'x - 1/2tx'\Sigma x \mid x'x = 1 \right\},
\end{equation}

where $T$ is the investor’s fixed risk tolerance parameter from (2) and the new PQP parameter $t$, through the term $tq$, captures the change in $\mu$, i.e., $\mu(t) = \mu + tq = \mu + \Delta \mu$.\(^{13}\) Best and Grauer (1991a), (1991b), (1992) showed that the optimal portfolio is

\begin{equation}
(17) \quad x(t) = \Sigma^{-1}t/c + T \left[ \Sigma^{-1} \left( \mu - t a/c \right) \right] + tT \left[ (q'\Sigma^{-1}t/c) \right].
\end{equation}

As in the traditional Markowitz analysis, (17) is a linear function of the parameter $t$, i.e., $x(t) + h_0 + t h_1$, but, in this case, the parameter captures a change in the means.

\(^{13}\)For example, the results of the computations reported in Table 2 are based on perturbations in the $(\Sigma, x^*)$-compatible means of either plus or minus 1, 2, 5, or 10 basis points. We can view these perturbations as random, or we can make the four perturbations consistent with (16) by first setting $q$ equal to a vector of plus or minus ones and then stopping the parameter $t$ at values of 0.0001, 0.0002, 0.0005, and 0.001, respectively.
The constant portion of the portfolio, $h_0$, is equal to $\Sigma^{-1} L/c + T(\Sigma^{-1}(\mu - L a/c))$, which from (6) and (7) is the solution to investor $T$’s MV problem with no change in the means, i.e., the complete solution to (2). Then, clearly, the $h_1$-vector, $T(\Sigma^{-1} q - \Sigma^{-1} (q' \Sigma^{-1} L)/c)$, gives the rate of change in investor $T$’s portfolio as a function of $t$. The fact that the change in the portfolio weights is a function of $\Sigma^{-1}$, which will most likely contain some large elements, suggests that portfolio composition could be extremely sensitive to changes in the means. On the other hand, an investor with no tolerance for risk would hold the (unchanged) global minimum-variance portfolio.

Further insight into the sensitivity of the optimal portfolio’s weights may be obtained in terms of upper bounds on the variation in $\mathbf{x}$ corresponding to the variation in the mean vector $\mu + tq$. Let $\|\mathbf{y}\|$ denote the Euclidean norm of the vector $\mathbf{y}$ and $\phi_{\min}$ and $\phi_{\max}$ be the minimum and maximum eigenvalues of $\Sigma$, respectively. Best and Grauer (1991b) showed that the bounds are derived from the definitions of $h_0$ and $h_1$ in (17). The bound for $h_0$ is

$$\|h_0\| \leq \frac{\phi_{\max}}{n^{1/2} \phi_{\min}} + \frac{T \|\mu\|}{\phi_{\min}} \left(1 + \frac{\phi_{\max}}{\phi_{\min}}\right).$$

The bounds for the change in investor $T$’s optimal portfolio weights are

$$\|\mathbf{x} - \mathbf{x}^*\| \leq t \|h_1\| \leq t \left[\frac{T \|q\|}{\phi_{\min}} \left(1 + \frac{\phi_{\max}}{\phi_{\min}}\right)\right],$$

where $\mathbf{x}^*$ denotes the original, or initial, value of the portfolio weights. Equation (18) again reveals that the changes in the optimal portfolio’s weights could be extremely large as they are functions of the ratio of the largest to the smallest eigenvalue as well as the reciprocal of the smallest eigenvalue. Furthermore, (18) suggests that the more assets there are in the universe, the more sensitive MV-efficient portfolios will be to changes in the means. For example, the minimum and maximum eigenvalues in the 10-asset universe considered below are $0.677 \times 10^{-3}$ and $0.323 \times 10^{-1}$, respectively, while in the 100-asset universe, the eigenvalues are $0.321 \times 10^{-4}$ and 0.282.

The expected return, $\mu_p = (\mu + tq)'x$, and the variance, $\sigma_p^2 = x' \Sigma x$, of the optimal portfolio are quadratic functions of $t$,

$$\mu_p = \alpha_0 + \alpha_1 t + \alpha_2 t^2, \quad \sigma_p^2 = \gamma_0 + \gamma_1 t + \gamma_2 t^2,$$

where $\alpha_0 = \mu' h_0 = a/c + T (b - a^2/c)$, $\alpha_1 = \mu' h_1 + q' h_0 = (q' \Sigma^{-1} L)/c + 2T [q' \Sigma^{-1} \mu - (q' \Sigma^{-1} L) a/c]$, $\alpha_2 = q' h_1 = T [q' \Sigma^{-1} q - (q' \Sigma^{-1} L)/c]^2/c$, $\gamma_0 = h_0' \Sigma h_0 = 1/c + T^2 [b - a^2/c]$, $\gamma_1 = 2h_0' \Sigma h_1 = 2T^2 [q' \Sigma^{-1} \mu - (q' \Sigma^{-1} L) a/c]$, $\gamma_2 = h_1' \Sigma h_1 = T \alpha_2$. 

From (8), $\alpha_0$ and $\gamma_0$ are the mean and variance of investor $T$’s original portfolio. Clearly then, $\alpha_1 t + \alpha_2 t^2$ and $\gamma_1 t + \gamma_2 t^2$ show how the mean and variance of the optimal portfolio change as a function of $t$. We call the optimal mean-variance path either investor $T$’s mean parameterized efficient frontier or his portfolio expansion path (PEP). Best and Grauer (1992) derived the following closed-form expression for the investor’s PEP,

$$
(\sigma_p^2 - T \mu_p - \delta_1)^2 = \delta_2 (T \sigma_p^2 + \mu_p - \delta_3),
$$

where the three constants $\delta_1$, $\delta_2$, and $\delta_3$ are

$$
\delta_1 = (\gamma_0 - T \alpha_0) + \frac{(T \alpha_1 - \gamma_1)(\alpha_1 + T \gamma_1)}{2\alpha_2 (1 + T^2)},
$$

$$
\delta_2 = \frac{(\gamma_1 - T \alpha_1)^2}{\alpha_2 (1 + T^2)} = \frac{T \left( (q' \Sigma^{-1} q)^2 \right)}{(1 + T)^2 c \left[ (q' \Sigma^{-1} q) - (q' \Sigma^{-1} l)^2 \right]},
$$

$$
\delta_3 = \frac{1}{\delta_2} \left[ -\delta_1^2 + (\gamma_0 - T \alpha_0)^2 + \frac{(\alpha_1 \gamma_0 - \gamma_1 \alpha_0)(T \alpha_1 - \gamma_1)}{\alpha_2} \right].
$$

The PEP is a parabola symmetric about the line

$$
\mu_p = -\frac{\delta_1}{T} + \frac{1}{T} \sigma_p^2.
$$

For every value of $t$ in (16)–(20), there is a new minimum-variance frontier corresponding to the new mean vector $\mu(t) = \mu + tq$ (and the covariance matrix $\Sigma$). Best and Grauer (1991b), (1992) contain figures showing two such minimum-variance frontiers and the PEPs of investors with different risk tolerance parameters. The implication of the analysis is that small changes in asset means will almost certainly cause large changes in the portfolio weights, but may cause either large or small changes in the mean and variance of the optimal portfolio.

III. The Computational Methodology

In the computational sections, we employ stock market return data and computational analysis to shed additional light on two of the questions raised earlier: First, with $(\Sigma, x^*)$-compatible means, which ensure the positively weighted portfolio $x^*$ is MV efficient, what proportion of the minimum-variance frontier contains positively weighted portfolios? Second, if we perturb the $(\Sigma, x^*)$-compatible mean vector, what (if any) portion of the new minimum-variance frontier contains

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14Suppose the positively weighted portfolio $x^*$ is MV efficient for the investor with risk tolerance parameter $T$ (or equivalently $r^*$). Comparing the efficient set analysis (2)–(9) with the sensitivity analysis (16)–(20), it is clear that, by specifying $q$, we could deduce how far any mean or subset of means could change before one of the portfolio weights would become negative. For example, using similar arguments to the ones employed in the analysis surrounding (10)–(12), we could show that the positively weighted portfolios are confined to a single segment of the investor’s PEP, a segment that corresponds to lower and upper bounds on the PEP parameter $t$ in (16). By imposing nonnegativity constraints on (16), Best and Grauer (1991b) provide estimates of the size of the shifts in the means that would drive from one up to one-half the assets from an $n$-asset universe.
positively weighted portfolios? We identify our investment universe as the 958 firms on the CRSP tape for which there are no missing rate of return data over the 120-month period from January 1976 to December 1985. From this universe of 958 securities, we select random samples (without replacement) of size \( n \), where \( n = 10, 20, 50, \) or 100 securities.

We address the first question by proceeding in three steps. First, we calculate a covariance matrix from historical data. Second, given the covariance matrix, we calculate a set of \((\Sigma, \theta^*)\)-compatible means from (1) and focus on one set of realistic \( \theta_1 \) and \( \theta_2 \) values that have been chosen to yield reasonable expected returns, expected excess returns, and risk tolerance parameters. \( \theta_1 \) was chosen so that the zero-beta rate is 0.75 percent per month (9 percent per annum) and \( \theta_2 \) was chosen so that the expected excess return on the optimal MV portfolio is also 0.75 percent per month. This latter value corresponds to the excess return of the Standard & Poor’s 500 Index over Treasury bills during the last half century. Having chosen \( \theta_2 \) in this way, it is clear from (13) that there will be an implied risk tolerance parameter for each data set. Under certain conditions, it can be shown\(^{15}\) that if (2) is used to approximate expected utility for an isoelastic function of the form,

\[
u(w) = \frac{1}{\gamma} w^\gamma, \quad \gamma < 1,
\]

then the risk tolerance parameter \( t \) is the reciprocal of the Pratt-Arrow measure of relative risk aversion (RRA), where \( \text{RRA} = -w u''(w)/u'(w) \). That is, \( t = 1/(1 - \gamma) \). The data yielded values of RRA ranging between 2.92 and 3.73, which is consistent with Friend and Blume’s (1975) estimate that RRA is equal to 2.

Third, given that \( \mu_p = \mu^* s \), it is obvious that portfolios containing all positive weights must have means that lie between the minimum and maximum asset means. Let

\[
\mu_{\text{min}} = \min \{ \mu_1, \mu_2, \ldots, \mu_n \}, \quad \text{and} \quad \mu_{\text{max}} = \max \{ \mu_1, \mu_2, \ldots, \mu_n \}.
\]

We define the proportion of the minimum-variance frontier that contains portfolios with all positive weights as the ratio of the difference between the maximum and minimum positively weighted minimum-variance portfolio means to the difference between the maximum and minimum asset means, i.e.,

\[
(\mu_{\text{max}} - \mu_{\text{min}}) / (\mu_{\text{max}} - \mu_{\text{min}}).
\]

While we have chosen \( \theta_2 \) to reflect realistic economic values, it is important to note that the proportion of the frontier containing all positive weights is nonetheless invariant to \( \theta_2 \).

\(^{15}\)Ohlson (1975) and Pulley (1981) developed the MV approximation to expected utility for short holding periods. Grauer (1986) related it to the parametric quadratic programming framework of Markowitz (1970) and Sharpe (1970). Furthermore, under certain conditions in continuous time, the MV approximation to expected utility for the isoelastic functions is exact, see e.g., Merton (1973). We also note that the approximation requires that returns be measured as units plus rates of return, which we did in the empirical work.

\(^{16}\)We outline the argument as follows. Recall from (14) that \( \hat{\xi}_1 \) is proportional to \( \theta_2 \), i.e., \( \hat{\xi}_1 = \theta_2 (\xi^* - \hat{\mu}_0) \). Then, it can be shown that the quantities used in calculating \( t_\xi \) and \( t_t \) in (10) and (11) are
We then employ (21) and the conditions from Theorem 1 in order to determine what proportion of the frontier contains positively weighted minimum-variance portfolios. We address the second question by perturbing the \((\Sigma, \mathbf{x}^*)\)-compatible mean vector either by randomly increasing or decreasing all the means by a given amount or by increasing one mean by a given percentage of itself. Then, we use the conditions from Theorem 1 to check whether there are any positively weighted portfolios on the new frontiers.

IV. The Computational Results

The first question addressed is: if there is a positively weighted MV-efficient portfolio, what proportion of the frontier contains positively weighted portfolios? An equally weighted portfolio (EWP) serves as our base case in generating the \((\Sigma, \mathbf{x}^*)\)-compatible means. That is, we set \(\mathbf{x}^* = 1/n\), where \(n\) is the number of assets in the universe. Then the evidence is consistent with the assertion that the proportion of the minimum-variance frontier containing all positively weighted portfolios is small and decreases as the number of assets in the universe increases. For example, Table 1 shows that as the universe expands from 10 to 100 assets, the proportion of the frontier containing portfolios with all positive weights shrinks from 0.4017 to 0.0451.

Table 1 also shows the results for two sum-of-the-digits weighted portfolios that are different from each other and from the equally weighted portfolio. The weights in the ascending and descending sum-of-the-digits portfolios are

\[
\left(1/N, 2/N, \ldots, n/N\right), \quad \text{and} \quad \left(n/N, (n - 1)/N, \ldots, 1/N\right),
\]

respectively, where the sum of the \(n\) integers is \(N = (n(n + 1))/2\). The table shows that the portion of the frontier that contains all positively weighted portfolios, when sum-of-the-digits portfolios are MV efficient, is smaller than for the corresponding EWP cases. For example, as little as 0.41 percent of one 100-asset frontier contains positively weighted portfolios with the ascending sum-of-the-digits weighting scheme.\(^{17}\)

Figure 4 plots the mean-standard deviation points for the 100-asset case along with two minimum-variance frontiers when the EWP is MV efficient. To avoid confusion, we call the minimum-variance frontier from (2) the unstrained minimum-variance frontier (even though the budget constraint, \(\mathbf{i}'\mathbf{x} = 1\), is active throughout). When nonnegativity constraints, i.e., \(\mathbf{x} \geq 0\), are imposed on (2), we call the minimum-variance frontier the nonnegatively weighted minimum-variance frontier. In Figure 4, the unconstrained minimum-variance frontier is a

\(\) proportional to \(1/\theta_2\). (Incidentally, this means that the asset whose weight is just zero at either \(t_2\) or \(t_4\) is the same for all values of \(\theta_2\).) It can also be shown that the difference \((\mu_{t_2} - \mu_{t_4}) = (t_2 - t_4)\beta_1\) is proportional to \(\theta_2\). Finally, from (1), \(\mu_{\max} - \mu_{\min}\) is proportional to \(\theta_2\). Therefore, the ratio \((\mu_{t_2} - \mu_{t_4})/(\mu_{\max} - \mu_{\min})\) is independent of \(\theta_2\). On the other hand, the shape of the frontier and changes in the weights in the minimum-variance portfolios for a given perturbation in the means are not the same for different sets of \((\Sigma, \mathbf{x}^*)\)-compatible means.

\(^{17}\)As tests of robustness, we drew an additional set of random samples of size 10, 20, 50, and 100 from the CRSP data set and examined a 20-beta ranked portfolio data set. The 20 portfolios contain virtually all the assets on the CRSP tape over a 540-month period. (See Best and Grauer (1985) for details.) We repeated the experiments for EWPs that were MV efficient in each of the data sets and found results consistent with the evidence presented in Table 1.
TABLE 1
The Proportion of Minimum-Variance Frontiers Containing Positively Weighted Portfolios

<table>
<thead>
<tr>
<th>Number of Assets in Universe</th>
<th>Equally Weighed</th>
<th>Ascending Sum-of-Digits Weighted</th>
<th>Descending Sum-of-Digits Weighted</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.4017</td>
<td>0.3947</td>
<td>0.3166</td>
</tr>
<tr>
<td>20</td>
<td>0.2323</td>
<td>0.1437</td>
<td>0.0507</td>
</tr>
<tr>
<td>50</td>
<td>0.0784</td>
<td>0.0386</td>
<td>0.0116</td>
</tr>
<tr>
<td>100</td>
<td>0.0451</td>
<td>0.0041</td>
<td>0.0105</td>
</tr>
</tbody>
</table>

1The proportion of a minimum-variance frontier containing positively weighted portfolios is defined as the difference between the maximum and minimum positively weighted minimum-variance portfolio means divided by the difference between the maximum and minimum asset means.

2For a frontier to contain any positively weighted portfolios, the means must be \((\Sigma, x^*)\)-compatible for some positively weighted portfolio, see (1). The ascending and descending sum-of-the-digits weights are defined in (22).

clearly defined hyperbola. Positively weighted MV-efficient portfolios are confined to the segment of the frontier between the two squares. The nonnegatively weighted minimum-variance frontier is a piecewise hyperbola. (See Best and Grauer (1990).) At the lower (upper) end it consists of the low (high) expected return asset, while the global minimum-variance portfolio consists of positions in 18 assets. The difference between the standard deviation of the global unconstrained and nonnegatively weighted minimum-variance portfolios is quite marked. (The respective standard deviations are approximately 0.98 and 2.6 percent.) In the interval between the two squares, roughly between means of 1.46 and 1.53 percent, the two frontiers coincide exactly. Furthermore, they remain very close to each other over a much wider range of means. Even so, for means below 1.3 or above 1.7 percent, portfolios on the nonnegatively weighted minimum-variance frontier contain less than 60 assets.

The second question addressed is: if we perturb the \((\Sigma, x^*)\)-compatible mean vector, what (if any) portion of the new frontier contains positively weighted portfolios? While every perturbation and every way of measuring its effect is arbitrary, there appear to be some more or less natural ways of perturbing the means and measuring a perturbation’s effect. At one extreme, we perturb all the means. To be more specific, we perturb all the means by either plus or minus 1, 2, 5, or 10 basis points. At the other extreme, we perturb just one mean.18 We measure a perturbation’s effect by reporting first whether there are any positively weighted minimum-variance portfolios on the new frontier and then by reporting the characteristics of the weights in four specific portfolios that correspond to the four ways of selecting an efficient portfolio. In the absence of a perturbation in the \((\Sigma, x^*)\)-compatible means, each of these methods would have selected the equally weighted MV-efficient portfolio. The four portfolios are labelled A, B, C, and D in Figure 5 and Tables 2 and 3 below. Portfolios A and B follow from the definitions of MV efficiency. That is, we find portfolio A (B) by minimizing (maximizing) the

18If the goal were to maximize the norm of the change in the weights, we would set \(g\), the rate of change in \(\mu\), equal to the eigenvector corresponding to the largest eigenvalue of \(\Sigma\).
variance (mean) given the mean (variance) of the EWP. We find portfolio C as the
tangency point by assuming that there is a constant riskless rate of 0.75 percent.
Finally, we find portfolio D by solving (2), given the risk tolerance parameter $T$.
Of the four, portfolios A and D are arguably the economically most interesting.
Portfolio A is noteworthy because it is the minimum-variance portfolio having
the highest correlation with the EWP. (In fact, the correlation is $\sigma(A)/\sigma(\text{EWP})$. See
Kandel and Stambaugh (1987).) On the other hand, portfolio D is a point on the
portfolio expansion path (PEP) of the investor with risk tolerance parameter $T$. (If
the reader were to interpret the examples as if the EWP were the market portfolio
in a CAPM setting, this investor would be the representative investor.) Then, for
portfolios A and D, we report: i) the number of negative weights; ii) the maximum
and minimum weights; and iii) the ratio of the norm in the change of the portfolio
weights to the norm in the change in the asset means. For portfolios B and C, we
simply report iii).
We consider simultaneous perturbations in all the means first. That is, we randomly perturb the $(\Sigma, x^\star)$-compatible means by either plus or minus 1, 2, 5, or 10 basis points. Perhaps surprisingly, the EWP has the same mean, 1.5 percent (or 150 basis points), and variance before and after the perturbations. With the exception of perturbations of either plus or minus one or two basis points in the means for a 20-asset universe, the new frontiers contained no positively weighted portfolios. Consider the minimum-variance portfolios with the same mean (portfolio A) or same risk tolerance parameter (portfolio D) as the EWP had before the perturbations in the means. Table 2 shows that, for the 50- and 100-asset universes with perturbations of either plus or minus one or two basis points, about half the weights in portfolio A were negative. Moreover, the magnitude of the weights was striking. For the 100-asset universe with a perturbation of plus or minus two basis points in the means, portfolio A’s weights ranged from $-1.119$ to $0.888$. Surprisingly, for larger perturbations in the means, the weights were less extreme. On the other hand, with perturbations of plus or minus 10 basis points, portfolio D had weights that ranged from $-7.6$ to $5.5$! Compare these weights to weights of 0.01 for the MV-efficient EWP portfolio before the perturbations in the means. Clearly, we are not dealing with a case of small differences in the means causing small changes in the weights. The changes in the weights are anything but small.

One might reasonably argue, however, that changing all the means, even if it is only by plus or minus one to 10 basis points each, becomes a large change when the number of assets increases. Therefore, it is meaningful to examine the change in the portfolio weights relative to the change in the asset means. We do so by taking the (Euclidian) norm of the change in the weights to the norm of the change in the means. Table 2 shows the weights are more sensitive to changes in the means as the number of assets increases. For example, with constant risk tolerance, the norm of the change in the weights is 176 (1697) times the norm of the change in the means in the 20- (100-) asset universe. Interestingly, for a given asset universe, this ratio is constant along the portfolio expansion path.

Turning from the weights to the frontier, Figure 5 depicts the 100-asset case when the $(\Sigma, x^\star)$-compatible means have been perturbed by plus or minus five basis points. Figure 5 has been drawn to the same scale as Figure 4 so that the figures may be easily compared. It is difficult to distinguish the difference in the asset means. However, the unconstrained minimum-variance frontier has shifted rather dramatically. One way to see this is in terms of the EWP. It is in exactly the same point in the two figures. The EWP is MV efficient in Figure 4. In Figure 5, the MV-efficient portfolio with the same mean as the EWP (portfolio A) has a standard deviation that is 0.54 times as large as the EWP’s. Furthermore, portfolio A contains 49 negative weights and has minimum and maximum weights of $-1.111$ and $0.975$, respectively. The MV-efficient portfolio with the same standard deviation as the EWP (portfolio B) has a mean of 2.14 percent, i.e., the mean is 1.43 times as large as the EWP’s. In addition, it contains 47 negative weights and has minimum and maximum portfolio weights of $-2.120$ and 1.653, respectively.

A second way to see how far the unconstrained minimum-variance frontier has shifted is in terms of the nonnegatively weighted minimum-variance frontier. Whereas the constrained and unconstrained frontiers coincided in a segment that


<table>
<thead>
<tr>
<th>Portfolio</th>
<th>( (\Sigma, x^*) ) ± 1 BP</th>
<th>( (\Sigma, x^*) ) ± 2 BP</th>
<th>( (\Sigma, x^*) ) ± 5 BP</th>
<th>( (\Sigma, x^*) ) ± 10 BP</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># of Neg Wts</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Min Wt</td>
<td>0.004</td>
<td>-0.037</td>
<td>-0.130</td>
<td>-0.178</td>
</tr>
<tr>
<td>Max Wt</td>
<td>0.101</td>
<td>0.153</td>
<td>0.295</td>
<td>0.450</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>175</td>
<td>172</td>
<td>157</td>
<td>122</td>
</tr>
<tr>
<td># of Neg Wts</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Min Wt</td>
<td>0.033</td>
<td>-0.045</td>
<td>-0.187</td>
<td>-0.424</td>
</tr>
<tr>
<td>Max Wt</td>
<td>0.101</td>
<td>0.151</td>
<td>0.303</td>
<td>0.555</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>176</td>
<td>174</td>
<td>176</td>
<td>177</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>176</td>
<td>174</td>
<td>166</td>
<td>145</td>
</tr>
</tbody>
</table>

Panel B: 50-Asset Universe

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>( (\Sigma, x^*) ) ± 1 BP</th>
<th>( (\Sigma, x^*) ) ± 2 BP</th>
<th>( (\Sigma, x^*) ) ± 5 BP</th>
<th>( (\Sigma, x^*) ) ± 10 BP</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># of Neg Wts</td>
<td>16</td>
<td>23</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>Min Wt</td>
<td>-0.066</td>
<td>-0.132</td>
<td>-0.202</td>
<td>-0.217</td>
</tr>
<tr>
<td>Max Wt</td>
<td>0.090</td>
<td>0.151</td>
<td>0.269</td>
<td>0.344</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>326</td>
<td>305</td>
<td>219</td>
<td>118</td>
</tr>
<tr>
<td># of Neg Wts</td>
<td>16</td>
<td>24</td>
<td>24</td>
<td>26</td>
</tr>
<tr>
<td>Min Wt</td>
<td>-0.077</td>
<td>-0.173</td>
<td>-0.464</td>
<td>-0.947</td>
</tr>
<tr>
<td>Max Wt</td>
<td>0.091</td>
<td>0.162</td>
<td>0.376</td>
<td>0.732</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>340</td>
<td>340</td>
<td>340</td>
<td>340</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>330</td>
<td>318</td>
<td>266</td>
<td>188</td>
</tr>
</tbody>
</table>

Panel C: 100-Asset Universe

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>( (\Sigma, x^*) ) ± 1 BP</th>
<th>( (\Sigma, x^*) ) ± 2 BP</th>
<th>( (\Sigma, x^*) ) ± 5 BP</th>
<th>( (\Sigma, x^*) ) ± 10 BP</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># of Neg Wts</td>
<td>44</td>
<td>46</td>
<td>49</td>
<td>53</td>
</tr>
<tr>
<td>Min Wt</td>
<td>-0.697</td>
<td>-1.119</td>
<td>-1.111</td>
<td>-0.673</td>
</tr>
<tr>
<td>Max Wt</td>
<td>0.541</td>
<td>0.888</td>
<td>0.975</td>
<td>0.697</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>1581</td>
<td>1267</td>
<td>517</td>
<td>171</td>
</tr>
<tr>
<td># of Neg Wts</td>
<td>42</td>
<td>46</td>
<td>46</td>
<td>46</td>
</tr>
<tr>
<td>Min Wt</td>
<td>-0.750</td>
<td>-1.511</td>
<td>-3.792</td>
<td>-7.595</td>
</tr>
<tr>
<td>Max Wt</td>
<td>0.564</td>
<td>1.118</td>
<td>2.779</td>
<td>5.548</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>1697</td>
<td>1697</td>
<td>1697</td>
<td>1697</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>1648</td>
<td>1485</td>
<td>955</td>
<td>541</td>
</tr>
<tr>
<td>Ratio ( \Delta Wts/\Delta \text{Mean} )</td>
<td>1309</td>
<td>1033</td>
<td>633</td>
<td>384</td>
</tr>
</tbody>
</table>

1The \( (\Sigma, x^*) \)-compatible means satisfy \( \mu = 0.75x + 0.25x^* \), where \( x^* \) is the equally weighted portfolio (EWP). With the perturbations, portfolio A (B) has the same mean (variance) as the EWP. Portfolio C is the tangency portfolio if the risk-free rate were 0.75 percent. The investor who held the EWP before the means were perturbed now holds portfolio D.

2Ratio \( \Delta Wts/\Delta \text{Mean} \) is the ratio of the Euclidean norm of the change in the portfolio weights to the Euclidean norm of the change in the means. For an \( n \)-asset universe, the norm of the change in the means for \( a \pm 1 \) basis point (BP) change is \( \sqrt{n} \times 0.0001 \). For changes of \( \pm 2, 5, \) or 10 BP, the norm is 2, 5, or 10 times as large.

3The \( (\Sigma, x^*) \)-compatible means in the 50- and 100-asset universes range between 0.95 and 2.23, and 0.94 and 2.36 percent, respectively.

contained the EWP in Figure 4, the nonnegatively weighted minimum-variance frontier plots well inside the unconstrained minimum-variance frontier in Figure 5. In the range of the EWP, the nonnegatively weighted minimum-variance portfolios contain 46 assets and have standard deviations that are very close to that of the EWP. Compare these standard deviations to portfolio A's, which, as noted, is 0.54 times the standard deviation of the EWP.

Now consider perturbing a single mean by, say, 10 or 50 percent of itself. Table 3 shows the characteristics of the weights in the four reference MV-efficient portfolios when selected \( (\Sigma, x^*) \)-compatible means are perturbed. Perhaps the most startling result in the table is for the 100-asset universe when the minimum \( (\Sigma, x^*) \)-compatible mean was perturbed from 0.94 to 1.03 percent. The MV-efficient portfolio with the same mean as the EWP had 54 negative weights, with minimum and maximum weights of -0.899 and 2.178, respectively, and the ratio of the norm of the change in the portfolio weights to the norm of the change in the asset means was 3434.19 Furthermore, each of these quantities was larger than

19Incidentally, the asset whose mean was increased was not necessarily the one with the largest weight.
the corresponding quantities when all the means were simultaneously increased or decreased by 10 basis points. For the other three portfolios, not every measure of the change was greater than when all the means were perturbed, but the changes were dramatic. For example, the ratio of the norm of the change in the portfolio weights to the norm of the change in the asset means ranged from 2625 to 4496.

Finally, we note that, in most of the cases examined here, small perturbations in the \((\sigma, \mathbf{x})\)-compatible means caused smaller perturbations in the minimum-variance frontiers than the one depicted in Figures 4 and 5. Yet, as the tables show, these frontiers contained no positively weighted portfolios. It is more than a little disconcerting that what may appear to be small, and perhaps unimportant, deviations from MV efficiency in return space (either mean-beta or mean-standard deviation space) are large and important deviations in portfolio policy (or weights) space.
TABLE 3
The Characteristics of the Weights of Four MV-Efficient Portfolios when Selected \((\Sigma, x^*)\)-Compatible Means are Perturbed

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whose Mean ((\Sigma, x^*)) Changes</td>
<td>Comp Mean (%)</td>
<td># of Neg Wts</td>
<td># of Neg Wts</td>
</tr>
<tr>
<td>min m&amp;v</td>
<td>0.95</td>
<td>14</td>
<td>-0.167</td>
</tr>
<tr>
<td>med m&amp;v</td>
<td>1.46</td>
<td>7</td>
<td>-0.020</td>
</tr>
<tr>
<td>max mean</td>
<td>2.23</td>
<td>8</td>
<td>-0.034</td>
</tr>
<tr>
<td>max var</td>
<td>1.97</td>
<td>11</td>
<td>-0.029</td>
</tr>
</tbody>
</table>

Panel A. 50-Asset Universe 10-Percent Increase in a Mean

<table>
<thead>
<tr>
<th>Panel B. 50-Asset Universe 50-Percent Increase in a Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>min m&amp;v</td>
</tr>
<tr>
<td>med m&amp;v</td>
</tr>
<tr>
<td>max mean</td>
</tr>
<tr>
<td>max var</td>
</tr>
</tbody>
</table>

Panel C. 100-Asset Universe 10-Percent Increase in a Mean

| min m&v | 0.94 | 54 | -0.899 | 2.178 | 3434 | 51 | -1.192 | 2.847 | 4496 | 3984 | 2625 |
| med m&v | 1.49 | 45 | -0.431 | 0.583 | 1121 | 45 | -0.539 | 0.684 | 1320 | 1205 | 2199 |
| max m&v | 2.36 | 50 | -0.393 | 0.953 | 1586 | 51 | -0.494 | 1.120 | 1916 | 1744 | 1709 |

See the notes to Table 2 for the definition of portfolios A, B, C, and D.

Key to abbreviations: min \(\equiv\) minimum, med \(\equiv\) median, max \(\equiv\) maximum, var \(\equiv\) variance, m&v \(\equiv\) mean and variance.

V. Summary and Concluding Remarks

Given any mean vector and a positive-definite covariance matrix, we derived simple, directly computable conditions for the resulting minimum-variance portfolios to have all positive weights. We showed that either there is no minimum-variance portfolio with nonnegative weights or there is a single segment of the minimum-variance frontier for which all portfolios have nonnegative weights. Then, we examined the likelihood of observing positively weighted minimum-variance portfolios. Analytical and computational evidence suggested that: i) even with \((\Sigma, x^*)\)-compatible means that ensure a positively weighted portfolio \(x^*\) is MV efficient, the proportion of the frontier containing positively weighted portfolios is small and decreases as the number of assets in the universe increases, and ii) small perturbations in the \((\Sigma, x^*)\)-compatible means result in minimum-variance frontiers containing no positively weighted portfolios. Clearly, the \((\Sigma, x^*)\)-compatible means in (1) impose quite a rigid structure on asset expected returns. We conclude by briefly noting some of the implications this structure has for portfolio management, for tests of the MV efficiency of a portfolio, for forecasting expected returns, for using the SML to measure investment performance, and for models of asset pricing.

First, consider the implications for portfolio management. CAPM and efficient markets arguments suggest that passive portfolio managers should hold the market portfolio. Even active managers are advised to simply shade out of or into over- and under-valued securities. Best and Grauer (1991b), studied the issue of diversification for a specific investor, employing the parametric quadratic program-
ming formulation of the sensitivity analysis for MV problems mentioned in Section II.C. The results indicated that, with short sales constraints, an MV decision maker, who believes that asset means are not almost exactly \((\Sigma, x^*)\)-compatible, will hold only about half the assets in a given universe, but the optimal portfolio’s expected return and variance will be virtually unchanged.

Second, consider the implications for tests of MV efficiency. A precondition for the CAPM to hold is that there are positively weighted minimum-variance portfolios: the central question is whether the positively weighted market portfolio is MV efficient. Grauer (1992) examined the power of multivariate tests of MV efficiency employing Monte Carlo simulation. Samples were drawn from populations where the hypothesis that the positively weighted market portfolio was MV efficient were true, and from populations where there were no positively weighted minimum-variance portfolios. The results showed that the multivariate tests of MV efficiency have very little power to reject the hypothesis that the market portfolio is MV efficient, even when there are no positively weighted minimum-variance portfolios in the population. For example, in a 20-asset universe, the population \((\Sigma, x_m)\)-compatible means were perturbed by larger amounts than any of the perturbations reported in this paper. Not surprisingly, there were no positively weighted minimum-variance portfolios in this new population. Yet, in this case, when the significance level of the test was set to 5 percent and samples of 120 observations on the 20 assets were taken, the multivariate test rejected the hypothesis that the market portfolio was mean-variance efficient only 8 percent of the time.

Third, consider the implications for forecasting expected returns and for using the SML to measure investment performance. Almost every textbook in corporate finance and in investments suggests the SML as a method of forecasting expected returns. The suggestion would appear to have considerable merit given Merton’s (1980) observation that variances and covariances are easier to estimate than means. Best and Grauer (1985) and Grauer (1991) turned the problem around somewhat and inferred the time series of means that were compatible with observed market value weights being MV efficient. We assumed that the covariance matrix was constant and inferred the \((\Sigma, x_m)\)-compatible means from the time series analog of (1),

\[
\mu_{mT} = \theta_{1T} + \theta_{2T} \sum x_{mT},
\]

where \(T\) is a time subscript. Essentially, this involves modeling the time series behavior of \(\theta_{1T}\) and \(\theta_{2T}\) by restricting, say, the zero-beta rate to be equal to the risk-free lending rate, by holding risk tolerance constant, by minimizing the change in the expected return vector from a non-time-varying vector of average returns, or by restricting various reward-to-risk measures.\textsuperscript{20} Ironically, in light of the sensitivity of the portfolio weights to any change in any one set of \((\Sigma, x_m)\)-compatible means, the time series behavior of the different sets of \((\Sigma, x_m)\)-compatible means were very different. Naturally, different time-varying SMLs, together with time-varying betas and different sets of time-varying means, add to the ambiguity inherent in using the SML to measure investment performance.

\textsuperscript{20}Merton (1980) also restricted reward-to-risk measures when forecasting the expected return on the market.
Finally, the evidence presented here in no way contradicts the theoretical result that, in a frictionless static Sharpe-Lintner world, the market portfolio is MV efficient. However, we must recognize that almost any deviation from either homogeneity of beliefs or MV preferences, the existence of differential taxes between dividends and capital gains, not to mention a multiperiod setting where prices continually adjust to new information, will mean that, at best, securities might plot close to the security market line. This might be perfectly acceptable for applications focusing on returns. But, if the evidence presented here is robust, close in return space will not be close in portfolio-weights space. It is highly unlikely that the market portfolio, or any other positively weighted portfolio for that matter, could be MV efficient unless the security market line relationship holds as a virtual identity.

Appendix

Green and Hollifield (1989) extended Green’s results to the case where the weights of minimum-variance portfolios lie between upper and lower bounds,

$$-K_n \leq x_i \leq K_n, \quad i = 1, \ldots, n.$$  

In this appendix, we show how the direct conditions are easily extended to this case. Suppose we would like to know when the weights of minimum-variance portfolios lie between upper and lower bounds,

$$L \leq x_i \leq H, \quad i = 1, \ldots, n,$$

which includes as a special case $L = -K_n$ and $H = K_n$. Analogous to (10) and (11), define

$$t^L_\ell = \max \{ (L - h_{0i})/h_{1i} \mid \text{all } i \text{ with } h_{1i} > 0 \},$$

and

$$t^L_u = \min \{ (L - h_{0i})/h_{1i} \mid \text{all } i \text{ with } h_{1i} < 0 \},$$

for the lower bound $L$. (Notice (10) and (11) are special cases with $L = 0$.) Similarly, for the upper bound $H$, define

$$t^H_\ell = \max \{ (H - h_{0i})/h_{1i} \mid \text{all } i \text{ with } h_{1i} < 0 \},$$

and

$$t^H_u = \min \{ (H - h_{0i})/h_{1i} \mid \text{all } i \text{ with } h_{1i} > 0 \}.$$

(Note that for the upper bound case the lower (upper) value of the parameter $t$ is determined in terms of negative (positive) $h_{1i}$-values—just the reverse of the lower bound case.) Finally, define

$$t^\dagger_\ell = \max \{ t^H_\ell, t^L_\ell \}, \quad \text{and} \quad t^\dagger_u = \min \{ t^H_u, t^L_u \}.$$  

Then we have the following theorem.

Theorem.

(a) There are minimum-variance portfolios in the interval, $L \leq x_i \leq H$, for
\( i = 1, \ldots, n \), if and only if \( t^\uparrow_i \leq t^\downarrow_i \), and \( L \leq h_{0i} \leq H \) for all \( i \) with \( h_{1i} = 0 \).

The minimum-variance portfolios are given by \( x(t) = h_0 + th_{1} \) for all \( t \) satisfying \( t^\downarrow_i \leq t \leq t^\uparrow_i \), and correspond to the segment of the minimum-variance frontier defined by \( \mu_{pv} \leq \mu_p \leq \mu_{pu} \).

(b) A necessary and sufficient condition that *some minimum-variance portfolio*, \( x(t) \), not be in the interval, \( L \leq x_i \leq H \), for \( i = 1, \ldots, n \), is that either i) \( t < t^\downarrow_i \) or \( t > t^\uparrow_i \), or ii) \( h_{1k} = 0 \) for some \( k \) with \( h_{0k} < L \) or \( h_{0k} > H \), or both.

(c) A necessary and sufficient condition that *no minimum-variance portfolio* be in the interval, \( L \leq x_i \leq H \), for \( i = 1, \ldots, n \), is neither \( t^\downarrow_i > t^\uparrow_i \), or \( h_{0k} < L \) or \( h_{0k} > H \) for some \( k \) with \( h_{1k} = 0 \), or both.

It is worth noting that these conditions may or may not be more restrictive than the conditions that all the weights are positive. For example, there are no minimum-variance portfolios in the interval \(-0.019 \leq x_i \leq 0.019\), for \( i = 1, \ldots, 100 \), when a sum-of-the-digits portfolio (see (22)) is MV efficient in a 100-asset universe.
References


_________. “The Efficient Set Mathematics when Mean-Variance Portfolio Problems are Subject to General Linear Constraints.” *Journal of Economics and Business*, 42 (May 1990), 105–120.


