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An Alternative Test of the Capital Asset Pricing Model: Reply

By Pao L. Cheng and Robert R. Grauer*

In our 1980 paper we tested the joint hypothesis that prices are determined by the mean-variance (MV) capital asset pricing model (CAPM) and that beliefs are stationary. By focusing on the Invariance Law of Prices we avoided the questionable practice of estimating *ex ante* expectations with *ex post* returns. Moreover, we circumvented the need to identify the “true” market portfolio and hence avoided the ambiguity, noted by Richard Roll (1977), in the traditional security market line (SML) tests of the same joint hypothesis. However, Stuart Turnbull and Ralph Winter (T-W) and Richard Sweeney point to a further inconsistency in the joint hypothesis, that they believe can be removed by relaxing the stationarity assumption. This new concern is fundamental in that it applies to all empirical tests which assume stationarity of the return distribution, whether they are simply tests of the CAPM or tests employing the CAPM. The concern would apply a fortiori to tests that assume stationary betas as well. Both comments also suggest that the *ad hoc* addition of a random error term to our Invariance Law equation and the subsequent statistical tests of it are unnecessary. We first address these two criticisms and then address some further criticisms raised separately by T-W and Sweeney.

1. *Ex Post* Returns are not Drawn from the *Ex Ante* Return Distribution

The single period CAPM presumes that, based on homogeneous beliefs with respect to an *ex ante* joint-return distribution, investors choose MV efficient portfolios.† Fur-

*Simon Fraser University. We thank Ken Collins, John Herzog, and Frederick Shen for their perceptive comments, but naturally we are responsible for any errors.

†Note that we are concerned with testing the standard single period two-fund CAPM and are not attempting to develop a test of more complicated multi-

fther, although the CAPM is a one-period model, it seems clear that what both theorists and empiricists have in mind is a multiperiod setting, where, period-by-period, investors make portfolio decisions in accordance with MV tastes. Although the assumption of stationarity is a virtual prerequisite for statistical testing, it is not itself the cause of the inconsistency in the joint hypothesis. To see this, we consider three types of stationarity implicit in the literature and the comments. This will enable us to clearly see the points of controversy between ourselves and Sweeney and T-W. Moreover, we can show that as we move from the most to the least restrictive case the inconsistency remains. *Strong stationarity* assumes a constant *ex ante* joint-return distribution and a constant risk-free rate. It is the implicit assumption of the security market line (SML) tests and implies that the market weights remain constant over time. *Weak stationarity* assumes a constant *ex ante* joint-return distribution, but a changing risk-free (or zero beta) rate. This is the assumption of our 1980 paper and it seems somewhat more realistic in that it allows the interest rate and prices to vary over time. *Nonstationarity* assumes that the *ex ante* return distribution and risk-free rate can both change.

To illustrate the inconsistency, assume that there are no capital transactions, that is, no new capital is raised and no dividends are paid, and for the moment, assume a constant *ex ante* joint-return distribution. Let \( p_{jt} \) be the value of security \( j \) at time \( t \); \( p_{jt, t+1} \) be the...
value of security $j$ given state $s$ occurs at time $t + 1$; $r_{j,t}^s$, be the ex ante return, defined as one plus the rate of return, on security $j$ given state $s$ occurs at $t + 1$; $R_{j,t+1}^s$ be the ex post return on security $j$ given state $s$ occurs at $t + 1$; $r_{j,t}$ be the risk-free return at $t$; $\bar{r}_{j,t}$ be the expected return on the zero beta portfolio at $t$; $x_{j,t}$, be the market value weight of security $j$ at $t$, that is, $p_{j,t} / \sum p_{j,k,t}$; $w_t$ be the total wealth in the market at $t$ ($= \sum_k p_{k,t}$ because borrowing must equal lending in equilibrium); and $w_t^{t+1}$ be the total wealth in the market given $s$ occurs at $t + 1$. Note that according to the traditional myopic, or period-by-period, application of the CAPM $x_{j,t}$ depends only on $r_{j,t}$ (or $\bar{r}_{j,t}$), $\mu$, and $\Sigma$. Likewise, $x_{j,t+1}$ depends only on $r_{j,t+1}$ (or $\bar{r}_{j,t+1}$), $\mu$ and $\Sigma$, and not on which state occurs at $t + 1$. (Hence, $x_{j,t+1}$ is not superscripted with an $s$.)

Now, the MV portfolio selection problem at time $t$ is of economic interest only if we assume that $\Sigma^{-1}$ exists and that at least two elements of the $\mu$ vector differ. This can happen only if the ex ante joint-return distribution is nondegenerate. That is, there is some $s$, $j$, $k$ combination such that: $r_{j,t}^s \neq r_{k,t}^s$. In the simplest formulation of the model, $r_{j,t}$ is also given. Assuming an equilibrium exists, at $t$ investors’ portfolio choices endogenously determine $r_{m,t}^s$, $x_{j,t}$, and $p_{j,t}$ for all $j, s$, where by definition $p_{j,t} = x_{j,t} w_t$.

We emphasize that the CAPM is a single period model. If we stay in the single period setting, there is no problem. At the end of the period, ex post returns are drawn from the nondegenerate ex ante joint return distribution. But, if we envision a second period, we run into serious trouble, which cannot be resolved by assuming nonstationary returns.

In the multiperiod scenario, at time $t + 1$, investors take the proceeds of their investments and rebalance their portfolios depending on their beliefs and opportunities, $\mu$, $\Sigma$, and $r_{j,t+1}$. It is clear that the simple rebalancing of their portfolios at $t + 1$ did not affect the aggregate wealth in the economy.

Therefore,

$$w_{t+1}^s = r_{m,t}^s w_t = R_{m,t+1}^s w_t,$$

and hence the ex post return on the market portfolio is drawn from its ex ante return distribution, that is, $R_{m,t+1}^s = r_{m,t}^s$.

However, the value of security $j$ at $t + 1$, is

$$p_{j,t+1}^s = x_{j,t+1} w_{t+1}^s = x_{j,t+1} R_{m,t+1}^s w_t,$$

where $x_{j,t+1}$ is determined in the equilibrating process at $t + 1$ by $\mu$, $\Sigma$, and $r_{j,t+1}$. By definition, the ex post return on security $j$ is

$$R_{j,t+1}^s = (p_{j,t+1}^s) / p_{j,t}^s.$$

Substituting for $p_{j,t+1}^s$ and $p_{j,t}$ in the previous expression gives

$$R_{j,t+1}^s = x_{j,t+1} R_{m,t+1}^s / x_{j,t}.$$

With strong stationarity this implies $R_{j,t+1}^s = R_{m,t+1}^s$, for all $j$. But, recall that the portfolio selection problem is degenerate unless $r_{j,t}^s \neq r_{k,t}^s$, for some $j$ and $k$, so clearly the ex post returns cannot be drawn from the ex ante return distribution.\footnote{Barr Rosenberg and James Ohlson (1976) noted that each security would yield the same ex post rate of return with portfolio separation and strong stationarity. Grauer (1978b) noted the lack of correspondence between ex post and ex ante returns with separation and either strong or weak stationarity. In this paper we are concerned solely with the MV model. But the inconsistency arises in any linear risk-tolerance CAPM for which the separation property holds. See Mark Rubinstein (1974), Roll (1973), and Grauer (1978a). Thus, Sweeney’s statement in the second paragraph of his fn. 3 is misleading.}

It is at this juncture that we ask whether the simple expedient of assuming a nonstationary ex ante joint-return distribution and a changing risk-free return will remedy the situation.\footnote{Clearly, if the nonstationarity case will not remedy the situation, then neither will the weak stationarity case.} Suppose, then, that we allow $r_{j,t+1}$, $\mu_{j,t+1}$, and $\Sigma_{j,t+1}$ to shift so that the $x_{j,t+1}$s change to satisfy

$$r_{j,t}^s = x_{j,t+1} R_{m,t+1}^s / x_{j,t}.$$
that is, so that the ex ante and ex post returns agree for each asset in each state. It may be possible to find a set of $x_{i,j,t+1}$s such that the above equation holds—for some particular state $s$. But the choice of $x_{j,t+1}$s in general will not be correct for some other state, say, $s'$. That is, $r_{j,t}$, given $t$, has two dimensions, $s$ and $j$, whereas $x_{j,t+1}$ has only one dimension, $j$. Moreover, even if it were possible to solve the problem by making the ex ante distribution nonstationary, it would be an extremely artificial solution. The ex ante return distribution at $t + 1$ would have to change in just the right way to ensure that the vector of ex post returns at $t + 1$ was drawn from the ex ante distribution seen at $t$.\(^6\) Furthermore, not only would it be an artificial solution, but the nonstationarities themselves would most likely rule out testing of a model whose primary appeal was potential testability.

To reconcile our results with T-W’s and Sweeney’s, note that there are two key conditions that should be met before the single period CAPM can be meaningfully embedded in a multiperiod setting: 1) the ex ante joint-return distribution must be nondegenerate, 2) the ex post returns must represent drawings from the ex ante return distribution. To their credit, both comments recognized an inconsistency in the joint hypothesis—Sweeney with strong stationarity and Turnbull-Winter with weak stationarity. However, neither comment recognized how basic the problem is—they both implicitly assumed condition 2) held. On the other hand, we showed the conditions 1) and 2) are inconsistent even in the most general case of nonstationarity. Thus, the relaxation of the stationarity assumption cannot overcome the contradiction of condition 2). These points are crucial. Unfortunately, Sweeney did not understand them. His new degeneracy arguments in Section I and footnotes 4–6 are incorrect because they are still based on the assumption that there is a correspondence between ex post and ex ante returns.

Simply stated, the CAPM is a single period model. Attempting to embed the single period CAPM in a multiperiod setting implies that the investors in such a model would have to behave in an inherently irrational manner basing their portfolio choices on ex ante beliefs that are never realized. The problem is not with the assumption of stationarity. Rather the problem is the lack of a multiperiod theory. While we were aware of these difficulties, we were also aware that the single period CAPM has provided many valuable insights into security pricing and has risen above a host of other valid criticisms. That is why we set our sights on a more manageable task: to provide an unambiguous test of the single period CAPM, without having to share Roll’s concern over the identity of the true market portfolio. Note that by focusing on prices we were also able to avoid the equally important issue of estimating ex ante returns with ex post data.\(^3\)

### II. The Need for a Test and the Missing Generating Model

Turnbull-Winter and Sweeney claim that once we developed the Invariance Law equation, there was no need to add an ad hoc error term in order to test it. This reflects a lack of appreciation for the process of modelling on their part, and perhaps a lack of a formal justification of the testing equation on our part.

The art of modelling is to capture the essence of reality in a simple, understandable way. It is naive to expect that the CAPM, or any other model for that matter, will hold exactly. The interesting question is whether the main implications of the model are upheld. If we get consistently statistically significant evidence that contradicts our hypotheses, then we may wish to question the model. On the other hand, even if the model does not hold exactly, if we cannot con-

\(^3\)Clearly the lack of correspondence between ex post and ex ante returns creates problems for any SML test whether it is a test of the MV SML or a test of the generalized SMLs generated from the linear risk tolerance CAPMs. Thus, the tests of Roll (1973) and Grauer (1978a) were flawed as well.
sitionally reject the major hypotheses, we may be willing to maintain the model as a working hypothesis. That is the spirit of our test.

It is well known that it takes at least seven assumptions to derive the MV CAPM, each of which violates to some degree the conditions in the real world. (See Michael Jensen, 1972.) Thus, the addition of the error term is not ad hoc. It is there because we are testing a model. We can simply assert that the error term captures these effects and performs the test.

We can provide a rigorous rationale for including an error term based on a generating process for prices. The embedding of our one-period valuation equations in a multiperiod setting as

\[ p_t = \gamma_t^{-1}(\Sigma_t^{-1}u_t - \tilde{r}_{t,t}\Sigma_t^{-1}q) \]

\[ = \gamma_t^{-1}(u_t - \tilde{r}_{t,t}v_t) \]

appears to have initiated much of the controversy at least partially, because we did not make explicit a data-generating process for observed prices, which we now do.

Let \( \tilde{r}_t \) be the ex ante random return vector during the \( t \)th period, that is, the interval between \( t-1 \) and \( t \). Given stationary \( \mu \) and \( \Sigma \), the random returns can be well defined by

\[ \tilde{r}_t = \mu + H\tilde{e}_t, \]

where \( E(\tilde{r}) = \mu, H \) is a square matrix such that \( HH' = \Sigma \) and \( \tilde{e}_t \) is the \( t \)th unit normal random vector with zero intratemporal and intertemporal covariance between assets.

The specification of \( \tilde{r}_t \) in (2) now obligates us to make the distinction between observed prices and equilibrium prices, which was, unfortunately, not explicitly and carefully made in our 1980 paper. At \( t-1 \), \( p_{t-1} \) is observed, that is, a realization of \( \tilde{p}_{t-1} \). Let \( P_{t-1} \) be a diagonal matrix where the diagonal elements correspond to the elements in the vector \( p_{t-1} \). Then the random returns \( \tilde{r}_t \) in (2) may be written as

\[ \tilde{r}_t = P_{t-1}^{-1}\tilde{p}_t = P_{t-1}^{-1}\tilde{p}_t + H\tilde{e}_t, \]

where \( \tilde{p}_t \) is the expectation of \( \tilde{p}_t \). Premultiplying (3) by \( P_{t-1} \), we get

\[ \tilde{p}_t = \tilde{p}_t + P_{t-1}H\tilde{e}_t, \]

which is the data-generating process implicit in the alternative test. That is, the future security prices to be observed are equal to their expectations plus a linear combination of zero mean error terms. With rational expectations, \( \tilde{p}_t \) may be taken as the equilibrium price vector, that is,

\[ \tilde{p}_t = \gamma_t^{-1}(\Sigma_t^{-1}\mu + \tilde{r}_{t,t}\Sigma_t^{-1}q). \]

By taking \( \tilde{p}_t \) as equilibrium prices, we shall now show that (4) and (5) above imply equation (16) of our 1980 paper, intended as a stochastic generalization for regression purposes.

Take the \( j \)th element of \( \tilde{p}_t \) in (4):

\[ \tilde{p}_{j,t} = \tilde{p}_{j,t} + p_{j,t-1}H_j'\tilde{e}_t, \]

where \( H_j' \) is the \( j \)th row of \( H \). Similarly, the \( i \)th and \( k \)th elements of \( \tilde{p}_t \) are

\[ \tilde{p}_{i,t} = \tilde{p}_{i,t} + p_{i,t-1}H_i'\tilde{e}_t, \]

\[ \tilde{p}_{k,t} = \tilde{p}_{k,t} + p_{k,t-1}H_k'\tilde{e}_t. \]

Since (5) implies

\[ \tilde{p}_{i,t} = b_{i,j,k}\tilde{p}_{j,t} + c_{i,j,k}\tilde{p}_{k,t}, \]

substituting for \( \tilde{p}_{i,t}, \tilde{p}_{j,t}, \) and \( \tilde{p}_{k,t} \) from (6),

\[ \text{the left panel of p. 663 should therefore be stricken.} \]

Equation (15) in Part B of the same section remains an equilibrium relation, but the firm values in (16) should be understood as observed values.

\[ \text{This is extensively demonstrated on p. 663 of our} \]

\[ \text{1980 paper. Note that (9) in this reply is equivalent to} \]

\[ \text{(11) there.} \]
(7) and (8), we get

\[
\tilde{p}_{i,t} = b_{i,j,k} \tilde{p}_{j,t} + c_{i,j,k} \tilde{p}_{k,t} + \left( p_{i,t-1} H'_{i,t} \tilde{e}_t - b_{i,j,k} p_{j,t-1} H'_{j,t} \tilde{e}_t - c_{i,j,k} p_{k,t-1} H'_{k,t} \tilde{e}_t \right)
\equiv b_{i,j,k} \tilde{p}_{j,t} + c_{i,j,k} \tilde{p}_{k,t} + \tilde{e}_{i,j,k,t},
\]

which is equivalent to the two-regressor case of equation (16) of our 1980 paper.\(^{12}\) Note that the current formulation has the advantage of showing explicitly the dependence of the error terms on the lagged values of all variables.\(^{13}\) The reader can also verify that (10) can be easily extended to contain more than two regressors as well as to deal with the aggregation of firm values, which justifies the grouping procedure used in our regressions.

In particular, note that one-period equilibrium prices do not possess a multivariate distribution, but the observed prices do via a chosen data generating process. The data-generating process we have made explicit in this reply makes it clear that the rank of observed asset prices must be at least two. Geometrically, for example, the equilibrium price vectors \(\tilde{p}_t\), \(t \in T\), in the security space are \(T\) points exactly on the two-dimensional subspace spanned by \(\Sigma^{-1} \mu\) and \(\Sigma^{-1} q\), but the observed price vectors \(\tilde{p}_t\) are points around the subspace. It follows that the random term in (16) of our 1980 paper is not \textit{ad hoc} and the subsequent statistical tests are not unnecessary.\(^{14}\)

\[III.\] Additional Responses to Sweeney

Sweeney rederives the Invariance Law in his equation (8) then makes a series of statements that indicates he does not really understand it. We quote but two:

The linear homogeneous relationship of \(p_i\) to \(p_j\) and \(p_k\) in (8) is the same that Cheng and Grauer refer to as the “Invariance Law of Relative Prices,” or their “Random March,” implying that a doubling of \(p_i\) and \(p_j\) means \(p_j\) must double if the market portfolio is to remain efficient, that is, if the \textit{CAPM} holds. [p. 1197]

… However, if (8) is to hold in all periods, under stationarity, this implies that the elements of \(\mu\) are identical as are the elements of \(\Sigma\) and that no relative price ever changes. Thus, Cheng and Grauer are really testing these latter extreme and uninteresting hypotheses, not the \textit{CAPM}. [p. 1199]

Sweeney’s confusion arises because he thinks that the Invariance Law holds only with strong stationarity.\(^{15}\) In that case, prices only differ by a scale factor over time. But clearly we were concerned with weak stationarity, where \(r_{j,t}\) (or \(\tilde{r}_{j,t}\)) changes over time but the \textit{ex ante} joint-return distribution is constant. Each time \(r_{j,t}\) (or \(\tilde{r}_{j,t}\)) changes a new set of market value weights and prices are generated as part of the equilibrating process. A doubling of one price most certainly does not imply all prices will double. Instead the Invariance Law states that, whatever the time-series of prices is, if you pick any three securities their (changing) prices will maintain the \textit{same} linear homogeneous relationship from equilibrium to equilibrium. The last two sentences of the quote are also clearly incorrect and for the same reason. Sweeney assumes strong stationarity —a constant \(r_{j,t}\) (or \(\tilde{r}_{j,t}\))—while our 1980 article, Cheng (1980), and this reply clearly envision weak stationarity.\(^{16}\) The illustration

\(^{12}\)The treatment of dividends may be explicitly incorporated in (2)–(10) without changing the structure of the error term in (10). We will provide demonstration upon request.

\(^{13}\)Since the components of \(\tilde{e}_{i,j,k,t}\) involve all variables with mixed signs, the problem of heteroskedasticity appears minor.

\(^{14}\)However, T-W did anticipate yet another test (in progress) which deals with three periods of observation on three security prices. We briefly mentioned this in fn. 13 of our 1980 paper.

\(^{15}\)Sweeney’s confusion can be traced to his equation (2) where his \(\lambda\) is not time subscripted to correspond with the changing zero beta expected return and market portfolio weights. In particular, it can be shown that \(\lambda_i = (a - \tilde{r}_{i,t} c)^{-1}\), where \(a = q \Sigma^{-1} \mu\) and \(c = q \Sigma^{-1} q\).

\(^{16}\)The last two sentences of the quote also reflect his failure to appreciate the idea that there is a lack of correspondence between \textit{ex post} and \textit{ex ante} returns in a period-by-period formulation of the \textit{CAPM}. (See our Section 1.) We also note that an unfortunate piece of
Sweeney uses to reinforce his argument bears out our suspicion. First, the investors’ indifference curves do not remain constant by virtue of the assumption of return stationarity. Since the expected utility of final wealth is a function of prices and prices can change even when \( \mu \) and \( \Sigma \) are stationary, the use of a constant set of indifference curves in his illustration serves no purpose. Second, under the assumption of weak stationarity, \( \tilde{r}_{z,t} \) is endogenous. Although a market portfolio exists, there may not be anybody holding it (see Cheng, 1980).

IV. Additional Responses to Turnbull and Winter

Turnbull-Winter raise five points. The first point, having to do with the need to perform a statistical test of the Invariance Law, was addressed in Section II.

Their second point is that the theoretical derivation of the CAPM\(^{17}\) does not require security prices be subject to only two sources of variation at each point in time; the CAPM is consistent, in particular, with a \( K \) factor arbitrage pricing model for any \( K \geq 1 \). This point is related to their fourth point, which is essentially empirical in nature. They argue that our test is ambiguous because it has no power of discrimination between the CAPM and a two-factor arbitrage pricing model (APM).

Factor models, in general, and the APM, in particular, have been closely associated with both the theoretical and empirical development of the MV CAPM. However, failure to make a clear distinction between the two models, on both the theoretical and the empirical levels, has led to confusion in the literature, including the misleading comments by T-W.

On the theoretical level, the APM and CAPM are different models that just happen to yield similar results. The MV results hold independently of the type of return distribution postulated. Thus, with a stationary ex ante return distribution, even if there were a \( K \) factor model, with \( K > 1 \), if the economy is populated by MV decision makers, at each point in time all securities will be priced in terms of two factors, the zero beta factor and the market factor (or alternatively, from the pricing equation (1), in terms of \( \gamma_r \) and \( \tilde{r}_{z,t} \)). In fact, the limiting case of a factor model is the \( n \)-asset case we have been considering, where each of the \( n \)-factors corresponds to the return on one of the \( n \)-assets. (See William Sharpe, 1977, p. 131.) Clearly, then, T-W’s second point is incorrect. On the theoretical level, MV decision making and a stationary ex ante return distribution implies that equilibrium prices are subject to only two sources of variation at each point in time.

Turning to the empirical side, it is true that our test has no power to discriminate between Stephen Ross’s two-factor model and the CAPM. It was not intended or designed for that purpose. We simply tested the MV CAPM plus the assumption of a stationary return distribution.\(^{18}\)

We would be in complete agreement with their third point having to do with an inconsistency in the joint hypothesis that states prices are determined period by period in an MV economy characterized by weak stationarity had they but noted that the inconsistency has to do with a lack of correspondence between ex ante returns and ex post realizations. Their failure to recognize this possibility leads to their fifth point. Turnbull-Winter suggest that any model and test procedure that is to be internally consistent must relax the stationarity assumption. However, as we demonstrated in Section I, relaxing the assumption of stationarity does not remove the lack of correspondence between ex ante and ex post returns that arises when the one-period CAPM is embedded in a multiperiod setting.

\(^{17}\)Implicitly we assume a period-by-period framework with constant \( \mu \) and \( \Sigma \).

\(^{18}\)Given that we were not testing a two-factor APM (and stationary returns), we leave it to the reader to decide whether he wishes to interpret our direct evidence against the MV CAPM as providing by implication evidence against the two-factor APM.
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