Normality, Solvency, and Portfolio Choice

Robert R. Grauer


Stable URL: http://links.jstor.org/sici?sici=0022-1090%28198609%2921%3A3C265%3ANSAPC%3E2.0.CO%3B2-N

Your use of the JSTOR archive indicates your acceptance of JSTOR’s Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR’s Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

*The Journal of Financial and Quantitative Analysis* is published by University of Washington School of Business Administration. Please contact the publisher for further permissions regarding the use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/uwash.html.

*The Journal of Financial and Quantitative Analysis*
©1986 University of Washington School of Business Administration

JSTOR and the JSTOR logo are trademarks of JSTOR, and are Registered in the U.S. Patent and Trademark Office. For more information on JSTOR contact jstor-info@umich.edu.

©2002 JSTOR
Normality, Solvency, and Portfolio Choice

Robert R. Grauer*

Abstract
This paper examines whether investors with power utility functions choose mean-variance (MV) efficient portfolios when returns are approximately normally distributed and there is borrowing or lending at a riskless interest rate. The results show that the unlevered portfolios of power utility investors plot very closely to the MV-efficient frontier. However, there are marked differences in the mix of risky assets, regardless of whether the portfolios are highly concentrated or widely diversified. Such differences allow power investors to remain solvent even when they lever their optimal portfolios to a greater extent than “less risk-averse” MV investors who risk bankruptcy. It is concluded that the investment policies of power utility and MV investors with similar risk aversion measures are not as similar as is commonly believed. This is particularly true for high power investors, unless explicit solvency constraints are imposed on the MV problem, and for low power investors when quadratic utility approximations are made to the power utility functions. These differences in the investment policies of power utility and MV investors lead us to question the widely-accepted assertion that the assumptions of homogeneous beliefs, normality, a riskless asset, and risk-averse investors imply the simple MV CAPM where all investors, including power utility investors, hold combinations of the market portfolio and the riskless asset.

I. Introduction

In the last thirty years, portfolio theory has thoroughly pervaded every aspect of finance. In that time, the expected utility and mean variance (MV) models have become the dominant forms of analysis. The expected utility model yields the most general decision-making framework, but only for those investors who can provide a detailed specification of tastes, represented by a specific utility function, and beliefs, described by a complete joint probability distribution. Even so, the model’s main disadvantage may be the computational problem. By way of contrast, the single-period MV model of Markowitz ([21], [22]) possesses an almost perfect balance between elegance and simplicity. Compared to the expected utility model, it offers a highly intuitive explanation for diversification without encountering the same information and computational problems.

* Department of Economics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6. Research support from the Natural Science and Engineering Research Council of Canada and the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged. The author thanks John Herzog for many helpful discussions, Nils Hakansson, and two anonymous *JFQA* referees for valuable comments, and Frederick Shen for most capable assistance. The author is responsible for any remaining errors.
However, the MV model does not explicitly consider some fundamental aspects of the investment problem addressed by other models. For example, the power or isoelastic utility functions given by

\[ u(w) = \frac{1}{\gamma} w^{\gamma}, \quad \gamma \leq 1, \]

where \( u(w) \) is the utility of wealth (which reduces to \( u(w) = \ln(w) \) when \( \gamma \) approaches 0), are of special interest because they possess a strong aversion to low returns and bankruptcy and are particularly applicable in multiperiod decision-making.\(^1\)

Quite naturally, considerable effort has been expended to determine the conditions under which an expected utility decision-maker, in general, and a power utility decision-maker, in particular, might pick from the MV-efficient set. It is well known that the MV model is consistent with the expected utility hypothesis when tastes are quadratic or returns are normally distributed.\(^2\) However, quadratic utility functions generate several implausible results, exhibiting negative marginal utility beyond some finite wealth level, as well as increasing absolute risk aversion. Therefore, normality provides a more popular way of reconciling the models. But, when the opportunity set consists of risky assets exhibiting an exact multivariate normal return distribution and a riskless asset, the only feasible solution for many, including anyone with tastes described by a power or log utility function, is to place all his capital in the riskless asset. The reason is simple. Any investor facing multivariate normal returns who places any wealth in the risky assets will find that his end-of-period wealth is also normally distributed, which implies that there is a positive probability of realizing a negative wealth level. However, the log and power functions are not defined over negative wealth levels and, hence, the expected utility does not exist.

These factors aside, there is a strong and almost overwhelming intuition that with approximate normality the small tail of probability extending to negative wealth levels can be "ignored," implying that the power utility and MV investment policies will be the same, i.e., all investors will hold the same portfolio of risky assets and will simply lever that portfolio up or down depending on their degree of risk aversion. On the other hand, there is an opposing intuition that suggests that, with approximately normal returns, the power utility functions' strong aversion to low returns and bankruptcy will lead them to pick portfolios that are not MV-efficient, i.e., the mix of risky assets for power utility investors will differ from that of MV investors.

This paper addresses the issue of which of these two intuitions is dominant. The primary concern is with opportunity sets in which returns are approximately normal, there is unlimited borrowing and lending at a riskless interest rate, and there are no short sales for risky assets, and with the issue of portfolio separation that provides the foundation for the traditional MV Capital Asset Pricing Model.

\(^1\) See, for example, [24], [12], [13], [17], [29], [9], and [14].

\(^2\) This paper is concerned exclusively with discrete-time models. It is well known that the assumption of a stochastic process of the Itô variety in continuous time also leads to MV decision-making (see [23]).
(CAPM) of Sharpe [31] and Lintner [19]. The study also is related to others that have examined the appropriateness of MV approximations to expected utility problems and whether investors with "similar" risk-aversion characteristics hold "similar" portfolios. The paper focuses on the issue of solvency, examines distributions in which MV investors, in particular, are widely diversified, and shows the relationship between one Taylor-series MV approximation to expected utility problems and the Parametric Quadratic Programming (PQP) approach to MV portfolio selection originated by Markowitz [22] and Sharpe [32].

The paper proceeds as follows. Section II relates the current study to others in the area. Section III formulates the investment problems. Section IV presents and interprets the results.

II. The Relation to Other Studies

Separation theorems play a central role in models of asset pricing. Cass and Stiglitz [4] identified the classes of utility functions and Ross [30] identified the classes of stochastic processes that permit separation. This paper is concerned primarily with the simplest two-fund separation theorem: in a world with homogeneous beliefs, a riskless asset, and normal returns, all investors, including those with power utility functions, choose portfolios that are combinations of the riskless asset and the same MV-efficient portfolio of risky assets. However, in relating this study to others in the area, it must be recognized that much of the literature that bears upon the question of whether those with power utility functions pick from the MV-efficient set did not examine opportunity sets containing normally-distributed returns and a riskless asset.

Hakansson [11] was the first to compare MV portfolios with the logarithmic utility (or growth-optimal) portfolio. Employing an example with two risky securities that exhibited nonsymmetric returns and a riskless asset, he showed that the growth-optimal portfolio was quite far from the MV-efficient frontier. Fama and MacBeth [6] took issue with the conclusions drawn from Hakansson’s work, asserting that with normality and a riskless asset the growth-optimal portfolio contains the same mix of risky assets as optimal MV portfolios, i.e., they ignored the problem of integrating a logarithmic utility function with a normal distribution. Then they proceeded to show that there is no difference in the ex post geometric mean of a specially-levered and an unlevered proxy for the market portfolio. Ziemb, Parkan, and Brooks-Hill [33] were able to solve the portfolio selection problem using MV analysis, numerical integration, exact normal distributions, a riskless asset, and power or logarithmic utility functions. However, they had to append linear segments to the utility functions in order to overcome the problem of integrability. On the other hand, Grauer ([7], [8]) focused on exact power functions by studying cases in which the opportunity sets contained approximately-normal returns generated from a discrete market model. Opportunity sets were also estimated from historical joint frequency distributions. The

3 Under the assumptions of homogeneous beliefs and identical opportunity sets for all investors that provide the foundations for the simplest MV model, the no-short-sales constraints on risky assets are not binding in market equilibrium. However, given that the separation property holds with or without these constraints, and policies that do not contain large short positions appear more reasonable in a real world setting, we explicitly include the constraints.
results showed marked differences in the mix of risky assets of MV and power utility investors. In contrast, there were small differences in the expected returns and standard deviations of the unlevered portfolios.

In closely related literature, Loistl [20], Levy and Markowitz [18], Pulley ([27], [28]), and Kroll, Levy, and Markowitz [16] investigated how closely portfolios picked on the basis of functions of means and variances can approximate portfolios picked by maximizing expected utility, while Kallberg and Ziemba [15] and Pulley considered whether investors with "similar" risk-aversion characteristics would hold "similar" portfolios.\(^{4}\) With the exception of Loistl, who discussed the difficulties associated with the use of Taylor-series approximations to expected utility problems (particularly logarithmic and power utility problems), the general consensus drawn from this literature is that portfolios picked on the basis of means and variances can closely approximate portfolios picked by maximizing expected utility, especially when the investors have similar risk-aversion characteristics.

### III. Formulation of the Investment Problems

#### A. The MV Model

The MV problem may be specified in two ways. It is usually formulated as a PQP problem in order to economize on the amount of input data and to ease the computational problem (see [22] or [32]). Let \( \mathbf{x}, \bar{\mathbf{x}}, \mathbf{t} \) be \( n \)-vectors containing the fractions of wealth invested in the \( n \)-risky assets, the means, and ones, respectively, \( x_{n+1} \) be the fraction of wealth invested in the riskless asset with return \( r \), and \( \Sigma \) be a \( (n,n) \)-covariance matrix, in which throughout any vector \( \mathbf{y} \) is assumed to be a column vector unless indicated to the contrary by transposition (e.g., \( \mathbf{y}' \)). Then, under the simplified but classic assumptions of unlimited borrowing or lending at the riskless rate, the MV investment problem is

\[
\text{max} \quad \left\{ \mathbf{t} \mathbf{\bar{x}}' \mathbf{x} + t r x_{n+1} - \frac{1}{2} \mathbf{x}' \Sigma \mathbf{x} \right\} \quad \mathbf{t}' \mathbf{x} + x_{n+1} = 1, \mathbf{x} \geq 0
\]

The efficient frontier is traced out as \( t \) varies from 0 to \( \infty \). (This holds whether or not there is a riskless asset and for general linear constraints.) Next it is noted that Pulley [27] employed a Taylor-series approximation to expected utility

\[
G[E(u)] = \bar{r}_p - \alpha \sigma^2_p,
\]

where \( G[E(u)] \) is the generalized expected utility approximation, \( \bar{r}_p \) is the mean rate of return on the portfolio, \( \sigma^2_p \) is the variance of the rate of return, and \( \alpha \) is equal to one-half the Pratt [26]-Arrow [1] relative risk-aversion coefficient (RRA)

\(^{4}\) Levy and Markowitz found that the orderings of individual assets by functions of mean and variance closely approximated expected utility orderings. However, they did not pick portfolios from these assets using either MV or expected utility methods. Kallberg and Ziemba employed normally-distributed asset returns and appended an exponential segment to the lower ends of the log and power functions in order to overcome the problem of integrability.
\( = -wu'(w)/u'(w) \) evaluated at a zero rate of return. The objective function in (2) is of the form

\[
q \mu_p - \frac{1}{2} \sigma_p^2,
\]

where \( \mu_p \) is unity plus the expected rate of return on the portfolio. Inspection of (2)–(4) shows that (2) provides an MV-approximation to expected utility problems with \( t = 1/RRA \). For a specific power utility function with \( RRA = 1 - \gamma \), the MV-approximation is given with \( t = 1/(1 - \gamma) \).

A second way to formulate the MV problem is in an expected utility framework employing quadratic utility. Let \( r_{js} \) be unity plus the rate of return on security \( j \) in the state \( s \) and \( \pi_s \) be the probability of occurrence of state \( s \). Then, the investment problem becomes

\[
\max \sum_s \pi_s \left\{ a_i \left[ \sum_{j=1}^{n} x_j (r_{js} - r) + r \right] - \left[ \sum_{j=1}^{n} x_j (r_{js} - r) + r \right]^2 \right\},
\]

subject to

\[
x_j \geq 0, \quad j = 1, \ldots, n,
\]

where the fraction of wealth borrowed or lent is given by \( x_{n+1} = 1 - \sum_{j=1}^{n} x_j \), and \( a_i \) is the measure of investor \( i \)'s risk aversion.

In general, a quadratic utility function is of the form \( u(w) = a^\gamma w - w^2 \), or it can be written in the return form used in (5) as \( u(r_p) = ar_p - r_p^2 \), where \( a = a^\gamma/w_0 \) and \( w_0 \) is initial wealth. Suppose we set \( w_0 = 1 \) and the return on the portfolio approaches zero, then a quadratic utility function will have the same degree of relative risk aversion as a power utility function if

\[
a = 2 + \frac{2}{1 - \gamma}.
\]

However, the approximation (5), with the value of the parameter \( a \) set from (7), must be approached with caution. For example, the indifference curves of an investor with quadratic utility are concentric circles centered at \( a/2 \) on the mean (vertical) axis in mean-standard deviation space. Moreover, beyond the value \( a/2 \), the quadratic utility function itself exhibits decreasing marginal utility. (See, for example, [32], p. 199.) Further, if \( a \) is set to a value less than twice the riskless return, \( r \), a quadratic utility function will attempt to lose money and will not choose portfolios along the MV-efficient frontier. In the empirical section, the policies of \(-10\) and \(-50\) power functions are examined, using both quarterly and annual decision horizons. But, it is easily verified that with quarterly returns and a riskless return of 1.025, the quadratic utility approximation to the \(-50\) power will not pick an MV-efficient portfolio. Moreover, for annual returns and a riskless return of 1.10, the quadratic utility approximation to the \(-10\) or \(-50\) power functions will not choose portfolios along the MV-efficient frontier.
B. The Power Utility Models

The power, or isoelastic utility functions, are less well known than the MV model. However, they have several noteworthy properties. First, they are consistent with multiperiod expected utility maximization whenever returns are independent over time (although independence is not required for the log function). Moreover, they are robust as investment objectives encompassing a wide variety of different tastes when the investment horizon is intermediate to long run. Second, they are myopic, i.e., independent of the perceived structure of returns beyond the current period. Third, they are the only class of functions for which it is legitimate to formulate the investment problem in rate of return as opposed to wealth form in a multiperiod setting. Fourth, they exhibit decreasing absolute risk aversion, which implies that the investor sees risky assets as normal goods. Fifth, they span a continuum of risk attitudes all the way from risk neutrality (γ = 1) to infinite risk aversion (γ = −∞). Moreover, the whole class, with the exception of γ = 1, displays an aversion to negative returns and bankruptcy that increases as γ decreases or, to put it in slightly different terms, the marginal utility of zero wealth is infinite.

The power utility investment problem itself is formulated as

$$\max \sum s \pi_s \frac{1}{\gamma} \left( \sum_{j=1}^{n} x_j (r_{js} - r) + r \right)^\gamma, \quad \gamma \leq 1,$$

subject to

$$x_j \geq 0, \quad j = 1, \ldots, n,$$

and

$$\sum_{j=1}^{n} x_j (r_{js} - r) + r \geq 0, \quad \text{for all } s.$$

The explicit solvency constraints (10) are not binding for the power functions, with γ < 1, and discrete probability distributions with a finite number of outcomes because of the functions' strong aversion to low levels of wealth or returns. Nonetheless, it is convenient to explicitly consider (10) so that the nonlinear programming algorithm ([2]) used to solve the investment problems does not attempt to evaluate an infeasible policy as it searches for the optimum. Moreover, (10) can be imposed on (2) or (5) to see whether it has any effect on the optimal MV policy.

C. The Generation of Multivariate Normal Returns

Having formulated the investment problems, the next step is to draw a sample from a multivariate normal return distribution that will serve as the ex ante joint return distribution. To do so, let \(Z\) be an \((n,m)\) matrix of \(m\) observations on an \(n\)-vector of normal random variables \(z\), i.e., \(z \sim N(\mu, \Sigma)\). Also, let \(Z\) be a

---

5 See footnote 1 for references.

6 This formulation follows Hakansson [10], Grauer [7], [8], and Grauer and Hakansson [9].
lower triangular matrix and \( I \) be an identity matrix, both of size \((n,n)\), and let \( y \) and \( 0 \) be \( n \)-vectors. Suppose that \( y \) contains independent normal variables with zero means and unit variances, i.e., \( y \sim N(0,I) \). Then, \( z = \mu + Ty \sim N(\mu, \Sigma) \), where \( \Sigma = TT' \). The \( m \) drawings from \( z \) constitute the columns of \( Z \).\footnote{More specifically, the International Mathematical and Statistical Library’s subroutines LUDECP was employed to find the lower triangular matrix and GGNML to generate pseudo-random normal \((0,1)\) deviates. Naylor, Balintfy, Burdick, and Chu [25] also suggest this way of generating a sample from a multivariate normal distribution. In this paper, losses on individual securities also were allowed to exceed 100 percent. While these losses occurred infrequently, the procedure differs from Grauer [7], [8] where losses were limited to 100 percent. On the other hand, the results show that unlevered portfolios do not risk losing anywhere near 100 percent.} The entries in a column correspond to the returns in a state that occurs with probability \( 1/m \). A typical entry in \( Z \) is \( r_{it} \). To repeat, \( Z \) is then taken to be the complete approximately normal \textit{ex ante} joint return distribution.

D. The Data

Quarterly and annual historical return data from ten individual securities and ten beta-ranked portfolios were used to provide realistic estimates of the mean vector and the covariance matrix. The data for the individual securities were adapted from [15] and the portfolio data were adapted from [5]. The data sets are quite different in that the individual securities are correlated on the order of 0.3 and the beta-ranked portfolios on the order of 0.9.

However, contrary to the predictions of the CAPM, investment policies based on historical data and subject to no short sales constraints on risky assets are usually not well diversified. Therefore, in order to more closely examine the micro foundations of an MV CAPM based on the assumptions of normality, a riskless asset, and risk-averse investors, we constructed approximately normal return distributions in which MV investors mix the proportion of their wealth committed to risk equally among the ten risky assets. We emphasize that there is nothing inherently special about an equally-weighted portfolio other than that it provides the simplest benchmark against which to detect any differences in the investment mix of well-diversified investors.

To be more specific, Best and Grauer [3] showed that, given \( \Sigma, \bar{x}^* \) is the optimal solution to the MV investment problem if \( \mu = \theta \mu + \theta_2 \Sigma \bar{x}^* \), where \( \theta_1 \) and \( \theta_2 \) are constants. Taking the covariance matrix from the joint normal sample, we can find an infinite number of \((\Sigma, \bar{x}^*)\)-compatible means. We experimented with two sets and report results based on the one that minimized the distance of the \((\Sigma, \bar{x}^*)\)-compatible means from the original means, keeping the riskless return constant, i.e., we set \( \theta_1 \) equal to the riskless return, and minimized the distance with respect to \( \theta_2 \). To construct the new joint return distribution, we simply shift the return on each security in each state by the difference in the two sets of means.

IV. Results, Interpretation, and Implications

A. Results

Table 1, panel A shows the investment policies for ten individual assets
based on quarterly data in which MV investors mix the proportion of wealth committed to risk equally among the ten risky assets. It contains the mix of risky assets for the unlevered portfolios, i.e., \( v_k = \frac{x_k}{\sum_{j=1}^{10} x_j} \), \( k = 1, \ldots, 10 \), and the sum of \( x_j \), i.e., \( \sum_{j=1}^{10} x_j \), which shows the total fraction of wealth committed to risk and implicitly allows the reader to calculate the fraction of wealth invested in each of the risky assets and the riskless asset.\(^8\) Panel B shows the return characteristics of these portfolios: the expected returns and standard deviations of the optimal levered and corresponding unlevered portfolios and the 1st, 5th, 10th, 50th, 90th, and 100th centiles of the return distributions of the optimal levered and corresponding unlevered portfolios. The assets have been ordered by expected return so that asset 1 (10) has the highest (lowest) expected return. (The same statement holds for the examples based on annual data.)

Panel B shows that the expected returns and standard deviations of the unlevered portfolios are virtually identical.\(^9\) This appears to support the almost universal intuition that with normality all investors pick from the MV-efficient set. However, panel A shows that the mix of risky assets differs among the investors with the mix of the 0.9 power investor differing most from the equally-weighted MV mix.\(^10\) More specifically, the 0.9 power investor invests nothing in assets 1 and 8, and invests at least 0.16 of the funds committed to risk in each of assets 4, 5, 6, and 10. It may also be noted that the investors have widely differing degrees of risk aversion and, hence, employ different degrees of leverage. For example, the 0.9 and \(-50\) power investors commit 6.95 and 0.04 times their wealth to risky assets.

The large differences in leverage and the seemingly smaller differences in the investment mix show up very clearly in the lower tails of the return distributions of both the optimally levered and corresponding unlevered portfolios. For each dollar invested in their levered portfolios, the 0.9, log, and \(-50\) power investors expect to get back 0.00 (actually 0.000008), 0.63, and 1.02, respectively, in the lowest return state (denoted as centile 1). By way of contrast, the lowest return on an individual security is 0.49.

The right-hand columns of Table 1 show two MV or quadratic utility approximations to the 0.9 power investor’s portfolio. The first MV approximation does not impose the solvency constraints (10) on the quadratic utility investor, while the second one does. (To the author’s knowledge, this is the first paper to impose explicit solvency constraints on MV investors.) From (7), a quadratic utility investor with an \( a_i \) of 22 and initial wealth of 1 will have the same relative risk aversion, 0.1, as the 0.9 power investor. This quadratic utility approxima-

---

\(^8\) The fraction of wealth invested in each risky asset is \( x_k = v_k (\sum_{j=1}^{10} x_j) \), \( k = 1, \ldots, n \). The fraction of wealth borrowed or lent is \( 1 - \sum_{j=1}^{10} x_j \).

\(^9\) The expected returns and standard deviations on the unlevered portfolios actually ranged from 1.0367 to 1.0362, and from 0.0798 to 0.0781, respectively. For the individual securities, the expected returns and standard deviations ranged from 1.0436 to 1.0315, and from 0.1787 to 0.0921, respectively. (With annual data, the expected returns of the ten securities ranged from 1.23 to 1.12, and the standard deviations from 0.55 to 0.13).

\(^10\) The differences between the true 0.9 power policy and the MV approximations to it are highlighted in the text only because the 0.9 power policy is the most different from the MV policy. It is clear from the results that the mix of risky assets is different for each of the power investors, and their mixes differ from the mix of MV investors. Sometimes the differences do not show up as clearly as they do at other times, but the MV-separation property still breaks down.
### TABLE 1
The Investment Policies and Return Characteristics of Selected Power and Mean Variance Investors Based on Quarterly Data

<table>
<thead>
<tr>
<th>Power</th>
<th>0.9</th>
<th>0.5</th>
<th>0.0</th>
<th>−10</th>
<th>−50</th>
<th>Mean Variance&lt;sup&gt;b&lt;/sup&gt;</th>
</tr>
</thead>
</table>

#### Panel A: The Investment Mix and Degree of Leverage<sup>c</sup>

<table>
<thead>
<tr>
<th>Asset</th>
<th>0.9</th>
<th>0.5</th>
<th>0.0</th>
<th>−10</th>
<th>−50</th>
<th>No Solv.</th>
<th>Solv.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.07</td>
<td>0.09</td>
<td>0.09</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.09</td>
</tr>
<tr>
<td>3</td>
<td>0.09</td>
<td>0.09</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.11</td>
</tr>
<tr>
<td>4</td>
<td>0.21</td>
<td>0.13</td>
<td>0.12</td>
<td>0.11</td>
<td>0.11</td>
<td>0.10</td>
<td>0.19</td>
</tr>
<tr>
<td>5</td>
<td>0.16</td>
<td>0.14</td>
<td>0.13</td>
<td>0.12</td>
<td>0.11</td>
<td>0.10</td>
<td>0.12</td>
</tr>
<tr>
<td>6</td>
<td>0.16</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.10</td>
<td>0.18</td>
</tr>
<tr>
<td>7</td>
<td>0.06</td>
<td>0.09</td>
<td>0.09</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.07</td>
</tr>
<tr>
<td>8</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.02</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.10</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.19</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
<td>0.24</td>
</tr>
<tr>
<td>Sum of x&lt;sub&gt;i&lt;/sub&gt;</td>
<td>6.95</td>
<td>3.61</td>
<td>1.92</td>
<td>0.18</td>
<td>0.04</td>
<td>18.65</td>
<td>7.17</td>
</tr>
</tbody>
</table>

#### Panel B: Descriptive Statistics, L-Levered, U-Unlevered<sup>d</sup>

| L   | U   | L   | U   | L   | U   | L   | U   | L   | U   | L   | U   | L   | U   | L   | U   | L   | U   | L   | U   | L   | U   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Mean 1.10 | 1.04 | 1.07 | 1.04 | 1.05 | 1.04 | 1.03 | 1.04 | 1.03 | 1.04 | 1.24 | 1.04 | 1.10 | 1.04 |      |
| Std. Dev. 0.55 | 0.08 | 0.28 | 0.08 | 0.15 | 0.08 | 0.01 | 0.08 | 0.00 | 0.08 | 1.45 | 0.08 | 0.57 | 0.08 |      |
| Centile 1 0.00 | 0.88 | 0.31 | 0.83 | 0.63 | 0.82 | 0.99 | 0.82 | 1.02 | 0.82 | −2.80 | 0.82 | 0.10 | 0.88 |      |
| 5 0.14 | 0.90 | 0.63 | 0.92 | 0.81 | 0.92 | 1.01 | 0.92 | 1.02 | 0.92 | −0.98 | 0.92 | 0.09 | 0.90 |      |
| 10 0.33 | 0.92 | 0.71 | 0.94 | 0.86 | 0.94 | 1.01 | 0.94 | 1.02 | 0.94 | −0.55 | 0.94 | 0.34 | 0.93 |      |
| 50 1.13 | 1.04 | 1.04 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.13 | 1.03 | 1.16 | 1.04 |      |
| 90 1.83 | 1.14 | 1.47 | 1.15 | 1.26 | 1.15 | 1.05 | 1.15 | 1.03 | 1.15 | 3.36 | 1.15 | 1.85 | 1.14 |      |
| 100 2.33 | 1.21 | 1.66 | 1.20 | 1.36 | 1.20 | 1.06 | 1.20 | 1.03 | 1.20 | 4.20 | 1.20 | 2.38 | 1.21 |      |

<sup>a</sup> The policies were selected from simulated approximately normal quarterly return data for ten individual securities. The means were adjusted so that the investment mix for an MV investor is 1/10th in each risky asset. The riskless return is 1.025. The assets are ordered according to expected return. Asset 1 (10) has the highest (lowest) expected return. All figures have been rounded to two decimal places.

<sup>b</sup> The MV solutions are for a quadratic utility function with an a<sub>i</sub> of 22 in (5). When wealth is scaled to 1, this investor has the same relative risk aversion, 0.1, as a 0.9 power investor. Solv. means that the solvency constraints (10) were imposed on (5).

<sup>c</sup> The investment mix is defined as x<sub>k</sub> = (x<sub>k</sub> / ∑<sub>j=1</sub>^10 x<sub>j</sub>), k = 1, . . ., 10. Sum of x<sub>i</sub> shows the fraction of wealth invested in risky assets. The x<sub>s</sub> were computed from (5) and (8), subject to the relevant constraints for the MV and power investors, respectively.

<sup>d</sup> The descriptive statistics show the expected value (Mean) and standard deviation (Std. Dev.) of the returns on the optimal levered and the corresponding unlevered portfolios. The selected centiles provide a more detailed description of the return distributions of the optimal portfolios. With the 100 state examples, centile 1 is the lowest return state.

The investment mix of the 0.9 power policy (without the solvency constraints) has an equal mix of risky assets with 18.65 times wealth invested in risky assets (as opposed to 6.95 times wealth for the 0.9 power investor). The expected return and standard deviation on the optimal levered portfolio are quite different from those of the 0.9 power investor, as well. More important, in the worst-return state, the quadratic investor risks losing 3.80 for each dollar invested. The table also shows that he risks bankruptcy in 10 of the 100 states. (He actually risks bankruptcy in 21 states.) However, the imposition of the solvency constraints on the quadratic utility investor alters the portfolio dramatically. He now invests 7.17 times his wealth in risky assets. Moreover, the investment mix and the return characteristics of the optimal portfolio are quite close to those of the 0.9 power investor.
Turning to the returns on the corresponding unlevered portfolios, it may be seen that the worst-state returns are higher for the less risk-averse power investors. While this might appear counterintuitive at first, the reason is quite straightforward. Basically, the less risk-averse power investors mix their risky assets in a manner different from MV investors so that they do not become bankrupt when the optimal amount of leverage is employed.

To save space, the investment policies for selected power and MV investors based on annual data are summarized here. The following points stand out. First, the investment mix of the power investors with annual data differs more from the equally-weighted MV mix than it does with quarterly data. Second, the investment mix of the less risk-averse power functions differs most from the MV mix. Third, higher power investors commit much less to risk with annual data. Fourth, the power investors risk losing more with annual data—but they do not risk bankruptcy. Fifth, the differences in the investment mix are quite dramatic for 10 beta-ranked portfolios. For example, while the MV investors invest equally among the 10 risky assets (portfolios), the log, $-10$, and $-50$ powers invest in only seven assets, with 37 or 38 percent of the funds committed to risk invested in asset 8. The 0.9 and 0.5 powers concentrate their investment in only 3 and 5 risky assets, respectively, with over 70 percent of the funds committed to risk invested in assets 4 and 7. Finally, none of the power investors commits anything to assets 5, 6, or 9.

B. Interpretation and Implications

In this section, the results are interpreted and related to others in the literature. First, it is noted that the PQP approximation (2) to power utility problems gave very similar results to the quadratic utility approximation (5), over the range of powers for which (5) is valid, and then the PQP formulation is employed as a framework for understanding a number of results found in this and other papers.

The MV-efficient frontier is traced out as $t$ varies from 0 to $\infty$ in (2). Suppose there are no riskless asset and binding constraints other than $\lambda^T x = 1$. Then it is well-known that the efficient frontier is a hyperbola in mean-standard deviation space. If linear constraints, say nonnegativity constraints, are imposed on the problem and they are binding, the efficient frontier becomes a series of piecewise hyperbolas. In the case of nonnegativity constraints and no riskless asset, as $t$ increases (relative risk aversion decreases) capital will be concentrated in the higher expected return-standard deviation assets. Thus, while a number of authors (see, for example, [28]) have reported that higher power investors hold only a small number of the higher expected return assets under these conditions, the results are not general. They follow from the specific formulation of the problem. When a riskless asset is added to the opportunity set, the efficient frontier is the line emanating from the riskless return and tangent to what was formerly the MV-efficient frontier of risky assets. If investors choose MV portfolios, they hold the same mix of risky assets. With approximate normality, we showed that the mix differs across MV and power utility investors. And, while the higher power investors hold fewer assets, our results clearly show that they are not the high expected return assets. Finally, when solvency and nonnegativity constraints are imposed on the problem, the efficient set is first a straight line, then it
becomes a series of piecewise hyperbolas with a "corner" portfolio (where the two vectors characterizing the MV-solution in an interval change), occurring as each of the constraints becomes active (or inactive).

Equation (2) is used to trace out the MV-efficient frontiers, with and without the solvency constraints, for the example presented in Table 1. The efficient frontiers are shown in Figure 1. With no solvency constraints, the traditional efficient frontier is the straight line passing through points A, B, C, and (beyond) D. The riskless asset plots at A, the unlevered MV policy at B, and the quadratic programming approximation to the 0.9 power policy at D. The solvency-constrained efficient frontier passes through points A, B, C, and E. The first solvency constraint becomes active at C when \( t = 2.61 \) (corresponding to a power function with a \( \gamma \) of 0.62). Before the MV-approximation to the 0.9 power with a \( t \) of 10 is reached, four solvency and four lower bound constraints are active. And by the time point E is reached, six solvency constraints have become active. The 0.9 power policy itself plots very closely to the solvency-constrained efficient frontier and to point E (at (0.5548, 1.1028) versus (0.5948, 1.1049) for E). The figure shows that the traditional MV approximation to the 0.9 power policy, point D, plots nowhere near the true policy, close to E. More generally, any solvency-constrained approximation to a power function, with a power higher than 0.62, plots closer to the true power policy than does the traditional MV-approximation in this example. However, it is emphasized that while these MV-approximations to the higher power functions, subject to the solvency constraints, represent better approximations to the true power policies, the data requirements are more onerous than for the traditional MV-analysis. The whole joint return distribution must be specified as constraints, i.e., the traditional MV model's near-perfect balance between elegance and simplicity is compromised on both the informational and computational levels.

Second, it is noted that the difference in the policies of high power and MV investors are more general than we have indicated. Two-fund separation theorems hold for the linear risk-tolerance utility functions, including the generalized power functions \( u(w) = 1/\gamma (w + b_i)^\gamma \), \( \gamma \leq 1 \), and the isoelastic functions, with \( b_i = 0 \), as special cases. \( (b_i) \) is a parameter that permits differing degrees of risk aversion for power investors sharing the same \( \gamma \) value. It is the analogue of the parameter \( a_i \) in (5) that permits differing degrees of risk aversion for quadratic utility investors.) The separation property states that investors with the same \( \gamma \) hold the same mix of risky assets independent of the level of initial wealth, \( w_0 \) and \( b_i \) (provided that \( w_0 > b_i/r \)). Thus, for a given \( \gamma \), an investor with a large positive \( b_i \)-value might risk losing over 100 percent of his capital, and another investor with a large negative \( b_i \)-value might be nowhere near risking 100 percent losses. However, both investors would mix their risky assets in exactly the same way as the isoelastic investor with \( b_i = 0 \). To be more specific, continuing the example from Table 1, if a 0.9 power investor had initial wealth of 1 and a \( b_i \) of \(-0.87 \), he would neither borrow nor lend. But he would still hold the same mix of risky assets as the 0.9 power investor with a \( b_i \) of zero, and not the equal mix of risky assets held by MV investors.

Third, some difficulties in measuring the expected utility loss caused by choosing an MV-approximation to the expected utility optimal portfolio are
The Traditional Efficient Frontier (ABCD) and the Solvency Constrained Efficient Frontier (ABCE).

noted. \( \xi \)roll, Levy, and Markowitz emphasized that Pulley’s comparisons based on ratios of expected utility, using MV-optimal weights, to expected utility, using utility optimal weights, were meaningless because the ratio could change completely depending on whether a constant was added to the utility function. It is somewhat ironic, then, that the index of utility loss based on a naive equally-weighted portfolio they proposed (see [16], p. 50) is simply arbitrary. In this paper, the examples were set up so that the optimal mix of risky assets for an MV investor is \( 1/n \) in each risky asset. Thus, if any unlevered power portfolio were compared with the unlevered MV portfolio, the proposed index would be zero. More importantly, the examples considered here showed that the traditional MV-approximations to the 0.9 power policy imply negative returns in many states. This means the MV policy is not feasible for the 0.9 power investor. In fact, the MV approximations are not feasible for any power investor with a \( \gamma \) of approximately 0.62 or greater in (1) in Table 1. Moreover, the same statement holds for even lower values of \( \gamma \) in the examples that employed annual data.

Finally, it is noted that the results appear to be both representative and robust. For example, the same types of differences in the policies occurred whether or not the means had been adjusted so that the MV investor would hold a widely-diversified portfolio or whether the opportunity set consisted of 100 or 200 states of nature.

Basically, the results indicate that the unlevered portfolios of power utility investors plot very close to the MV-efficient frontier. But there are marked differences in the mix of risky assets, whether the portfolios are highly concentrated or widely diversified. Such differences allow power investors to avoid the low
returns and risks of bankruptcy incurred by their MV counterparts. Therefore, it is
concluded that the investment policies of power and MV investors with similar
risk aversion measures are not as similar as is commonly believed. This is particu-
larly true for high power investors, unless explicit solvency constraints are im-
posed on the MV-problem, and for low power investors when quadratic utility
approximations are made to the power functions. Equally important, these differ-
ences in the investment policies of the MV and power utility investors lead us to
question the widely accepted assertion that the assumptions of homogeneous be-
iefs, normality, a riskless asset, and risk-averse investors imply the simple MV
CAPM in which all investors, including power utility investors, hold combina-
tions of the market portfolio and the riskless asset.

References

[2] Best, M. J. "A Feasible Conjugate Direction Method to Solve Linearly Constrained Optimiza-
Finance, 29 (June 1974), 857-885.
[11] _________________. "Capital Growth and the Mean-Variance Approach to Portfolio Selec-
[12] _________________. "On Optimal Myopic Portfolio Policies, with and without Serial Cor-
[13] _________________. "Convergence to Isoelastic Utility and Policy in Multi-period Portfolio
[16] Kroll, Y.; H. Levy; and H. M. Markowitz. "Mean-Variance versus Direct Utility Maximiza-


